Seminar on Shimura varieties and their canonical models

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1 Motivation

Shimura varieties are (families of) algebraic varieties that play an outstanding role in algebraic geometry and number theory. Because of Deligne's axiomatization of their theory in terms of so called Shimura data, it might be difficult at first to parse from the bare definitions why Shimura varieties are so interesting. Some of the main reasons are the following:

- Coming from the theory of moduli spaces, Shimura varieties generalize moduli spaces of elliptic curves and abelian varieties. In fact, the prototypical and easiest example of a Shimura variety is the modular curve, which parametrizes elliptic curves with certain level structures. The modular curve is of utmost importance in number theory (one instance being the now fully proven modularity conjecture, which led to the proof of Fermat's Last Theorem). This explains why we would want to also understand its higher-dimensional analogues - namely Shimura varieties.

- Shimura varieties are also important in the study of modular forms and their generalizations: Classically modular forms are defined as holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying certain growth and periodicity conditions that are dictated by an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$. From another viewpoint modular forms can be found in certain cohomology classes attached to the modular curve. Analogously other variants of modular forms (e.g. Hilbert modular forms, Siegel modular forms, etc.) can be found in the cohomology classes of higher-dimensional Shimura varieties.

- Shimura varieties and their cohomology groups are an important testbed for the Langlands program. In **very** broad terms, the cohomology groups of Shimura varieties naturally come equipped with an action by the adelic points of certain reductive groups, yielding representations of these adelic groups against which various Langlands-style conjectures can be tested.

- The theory of Shimura varieties uses many different branches of math. Learning about them will be an opportunity to learn about abelian varieties, reductive groups, some differential geometry of complex manifolds, some Hodge theory, and much more!

The plan for this seminar is to introduce the theory of Shimura varieties and their canonical models. If time and the number of participants permits, in the end we will also sketch various applications of the theory of Shimura varieties. Things I have in mind are the modularity theorem or the Eichler-Shimura congruence relations.

2 Prerequisites

As for prerequisites, some familiarity with algebraic geometry and the concept of moduli spaces will be important (however we will try to avoid using scheme language where possible, so classical algebraic geometry suffices). Basic differential geometry of complex manifolds will also be necessary (e.g. definition of complex manifold and differential forms).

Helpful (but not strictly necessary since we will review it) is some background on elliptic curves (or even abelian varieties) and on the theory of affine algebraic groups (specifically reductive groups). If you want to learn more about the latter, there will be a graduate seminar in the summer semester on the structure theory of affine algebraic groups, run by Prof. Tasho Kaletha.

3 Overview of the talks

- **Talk 0: Introduction and overview (April 7th)**

- **Talk 1: Abelian varieties and complex tori (April 14th)** First recall the general notion of abelian varieties, isogenies, polarizations, etc.

  Describe the classification of complex abelian varieties in terms of complex tori and Riemann forms.

  Then discuss abelian varieties over finite fields. Introduce Frobenius, Verschiebung, Weil q-numbers and the Honda-Tate theorem.

- **Talk 2: Miscellanea on algebraic and Lie groups (April 21st)**

  Quickly recall the notions of algebraic groups (classically) and Lie groups (main example: isometry groups of certain "symmetric spaces"), and their connection.

  Discuss the notions of reductive, semisimple, simply connected, derived, adjoint algebraic/Lie groups. Discuss the isogeny classification of semisimple groups, use it to describe compact/non-compact algebraic groups.

  Reductivity is useful for: Prop. 2.1, 2.2., beginning of chapter 5.

The real points of algebraic groups: Prop. 5.1, Thm 5.2, Cor. 5.3.

Discuss Cartan involutions on algebraic groups: Theorem 1.16, Example 1.17, Prop. 1.18, Prop 1.12.

Approximation theorems: Strong approximation (Thm 4.16), real approximation (Thm 5.4)

Explain the concept of Weil restriction via the example of the Deligne torus.

- **Talk 3: Hermitian symmetric domains and their classification (April 28th)**

  Introduce the notions of Hermitian manifolds, symmetric spaces and Hermitian symmetric spaces. Discuss their classification and define Hermitian symmetric domains. Cover Example 1.1a) and 1.2 from [Mil17]. Cover the subsection "Automorphisms of a Hermitian symmetric domain" (stress that Lemma 1.5 yields a classification of symmetric spaces as approximately being Lie groups modulo compact subgroups). Sketch the proof of Theorem 1.9 (you can freely use blackboxes. Focus on the use of the exponential map).

  The main result of this talk is Theorem 1.21. Sketch a proof. Mention the similarity to the definition of Shimura data we saw in Talk 0. Cover Corollary 1.22 and Example 1.23.

- **Talk 4: Variations of Hodge structures (May 5th)**

  Cover chapter 2 of [Mil17], more specifically: Briefly review Grassmannians and flag varieties (classically, no scheme language). Introduce Hodge structures and Hodge filtrations, discuss their connection. Then reinterpret Hodge structures as representations of the Deligne torus. Discuss the Weil operator, Hodge tensors and polarizations of Hodge structures.

  After introducing variations of Hodge structures, the main result of this talk will be Theorem 2.14. Sketch as much of the proof as time permits. Keep almost complex structures as a recurring example throughout the talk.

- **Talk 5: Arithmetic varieties (May 12th)**

  Cover chapter 3 of [Mil17], more specifically: Give the proof of Proposition 3.1 (explicitly state MF, 2.5). Define subgroups of finite covolume and arithmetic subgroups. State Proposition 3.2, Theorem 3.3 and Proposition 3.5. Prove Proposition 3.6. Mention that for our purposes, the Lie group H will be $\mathrm{Hol}(D)^+$ for a Hermitian symmetric domain D (which is adjoint).

  Recall Chow's theorem and then state the Theorem of Baily and Borel; mention the proof idea at least for $\mathcal{H}_1$. Define arithmetic varieties. State Theorem 3.14 and derive from it Corollary 3.16.

  Finally, state the Borel density theorem and show as much of the proof of Theorem 3.21 as possible.

- **Talk 6: (Connected) Shimura data (May 19th)**

  Cover the following from chapters 4 and 5 of [Mil17]: State the definition of connected Shimura data, Example 4.5 and Remark 4.6. Use Lemma 4.7 to prove the equivalence with the alternative definition of connected Shimura data.

  State Definition 5.5 and compare it to the definition of connected Shimura data via the discussion at the end of chapter 4. Compare example 4.5 with 5.6.

  To show how a Shimura datum is a "disjoint union" of connected Shimura data, discuss Proposition 4.9, Proposition 5.7 and Corollary 5.8. Finish by proving Proposition 5.9, clarifying how Shimura data are connected to variations of Hodge structures.

- **Talk 7: Shimura varieties (May 26th)**

  Start off by defining connected Shimura varieties as in Definition 4.10, mention Remark 4.11b) and Proposition 4.12. Discuss Example 4.14. Also mention the discussion at the beginning of chapter 5.

  Next, cover the subsection on congruence subgroups and finite adeles from chapter 4, focus on Proposition 4.1. Then prove Proposition 4.18 and 4.19 (blackbox 4.20), reinterpreting connected Shimura varieties (at least for G simply connected) as double coset space. Mention Aside 4.21a).

  Finally, define Shimura varieties as in Definition 5.14. Combine Lemma 5.11, 5.12, 5.13 and the discussion before Definition 5.14 to show that Shimura varieties are disjoint unions of arithmetic varieties. If there is time left, you can also mention Theorem 5.17 and the description of zero-dimensional Shimura varieties.

- **Talk 8: Example: Siegel modular varieties (June 2nd)** First discuss the additional axioms for Shimura data from chapter 5 of Milne's notes and mention the criterion Theorem 5.26 for SV5 to hold.

  Then cover chapter 6, more specifically the following subsections: Discuss the dictionary and introduce the notion of symplectic spaces. Introduce the Shimura datum attached to a symplectic space and show that it satifies SV1-SV6.

  Then describe the Siegel modular variety. You can use the results about complex abelian varieties from talk 1, but reinterpret Riemann's Theorem 6.8 in terms of Hodge structures. Using this, describe the moduli interpretation of Siegel modular varieties.

- **Talk 9: Hodge-type and PEL-type Shimura varieties (June 16th)** From chapter 7 of Milne's notes, introduce Hodge-type Shimura data and focus on Proposition 7.3 and Theorem 7.4. You can omit the discussion of the Hodge conjecture.

Then introduce PEL Shimura data and Shimura varieties as in chapter 8. More specifically, cover semisimple algebras, trace map, involutions, the different types of semisimple algebras and symplectic modules. An overview here is enough. Treat Example 8.6 so that we get used to the terminology. Probably it is best to use Proposition 8.7 as a blackbox.

Aim for the definition of PEL datum and PEL-type Shimura varieties. Most important are Proposition 8.14 and Theorem 8.17. It's okay if we don't see all the proofs.

- **Talk 10: Introduction to complex multiplication (June 23rd)** Cover chapter 10 from Milne's notes, more specifically: explain the program outlined at the beginning of the chapter (this motivation is very important!). Then introduce CM fields and abelian varieties of CM-type (the basics on general abelian varieties have been discussed in talk 1). Cover 10.2, 10.3 and 10.5.

  Since abelian varieties over finite fields were already discussed in talk 1, skip straight to Lemma 10.9 and the Taniyama-Shimura formula (the heuristics of Remark 10.11 suffice for us, you don't have to show the formal proof).

  Finish by discussing the $\mathcal{O}_E$-structure of the tangent space and stating Proposition 10.12

- **Talk 11: The main theorem of complex multiplication, special points (June 30th)** Start off by recalling the necessary amount of class field theory as in the beginning of chapter 11. Then introduce the notions of reflex field and reflex norm of a CM-type. State the main theorem of complex multiplication (and prove as much of it as possible).

  Then continue to put these notions into a broader context: define the reflex field as in chapter 12, cover Lemma 12.1, discuss Remark 12.3. Example 12.4a) and b) (and possibly c)) are also interesting to us.

  Introduce the notion of special points (12.5), mention 12.6 and via Example 12.7 (and the comment afterwards) relate them to things we already know.

- **Talk 12: Canonical models of Shimura varieties (July 7th)** Introduce the homomorphism r_x as in chapter 12 and discuss the definition of canonical models of Shimura varieties.

  Discuss a rough idea of how to prove uniqueness and existence of canonical models.

- **Talk 13: The Kottwitz conjecture and applications (July 14th)** This is very tentative, but I plan on taking section 3.2 from Morel's lecture notes on Shimura varieties as a basis for this talk.

References

[Mil17] J.S. Milne: Introduction to Shimura Varieties. Available online under `https://www.jmilne.org/math/xnotes/svi.html`.