TALK 2: STABLE ∞-CATEGORIES AND t-STRUCTURES

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1. Introduction

In this talk we will define t-structures and develop some of their associated theory. We will give examples and compute their hearts.

The goal of this talk is to give an exposition of the behaviour of stable ∞-categories and showcase the techniques available in this setting. Hence we have chosen to focus on proofs that make use of the stable structure. A slogan for this talk would be "kill it with exact sequences".

There are strong parallels between stable ∞-categories and triangulated (one)-categories, this will be addressed momentarily.

2. Recollections

We recall:

Definition 1. A pointed ∞-category $\mathcal{C}$ is **stable** if:

(a) $\mathcal{C}$ admits finite limits and colimits.

(b) A square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout if and only if it is a pullback.

Furthermore we also recall:

Fact 1. Let $\mathcal{C}$ be a stable ∞-category, then $h\mathcal{C}$ canonically has the structure of a triangulated category where the shift functor is given by $\Sigma : h\mathcal{C} \to h\mathcal{C}$ and a diagram $X \to Y \to Z$ is a triangle in $h\mathcal{C}$ if and only if it is a fibre sequence in $\mathcal{C}$.

Notation 1. Let $X, Y \in \mathcal{C}$ a stable ∞-category. Then we define

$$\mathrm{Ext}^n_{\mathcal{C}}(X, Y) := \pi_0 \mathrm{Map}_{\mathcal{C}}(\Omega^n X, Y)$$

and note for n negative this can be identified with $\pi_{-n}\mathrm{Map}_{\mathcal{C}}(X, Y)$. More generally, $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ can be identified with the $(-n)th$ homotopy group of the mapping spectrum from X to Y. It then follows from the definitions that given a fibre sequence $X \to Y \to Z$, for all $W \in \mathcal{C}$ we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{C}}(Z,W) \to \mathrm{Ext}^n_{\mathcal{C}}(Y,W) \to \mathrm{Ext}^n_{\mathcal{C}}(X,W) \to \mathrm{Ext}^{n+1}_{\mathcal{C}}(Z,W) \to \dots$$

of abelian groups.

3. Definitions and Basic Facts

We are now ready to define our object of interest:

Definition 2. Let $\mathcal{D}$ be a triangulated category, then a ***t*-structure** on $\mathcal{D}$ is the data of two full subcategories $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$, each stable under isomorphism such that:
(t_1) for $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq 0}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$,
(t_2) there are inclusions $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$,
(t_3) $\forall X \in \mathcal{D}$ there exists a fibre sequence $X' \to X \to X''$ where $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq 0}[-1]$.

Furthermore we define $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$.

Morally we can think of $(t1)$ as saying "$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq -1}) = 0$" and $(t3)$ as asserting there exists an exact sequence "$\mathcal{D}_{\geq 0} \to \mathcal{D} \to \mathcal{D}_{\leq -1}$".

Remark 1. Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ be a t-structure. Then $\mathcal{D}_{\geq 0}$ is uniquely determined as the full subcategory of $\mathcal{D}$ spanned by the objects X such that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \; \forall Y \in \mathcal{D}_{\leq -1}$. Similarly $\mathcal{D}_{\leq 0}$ is determined by $\mathcal{D}_{\geq 0}$.

Definition 3. A ***t*-structure** on a stable ∞-category $\mathcal{C}$ is a t-structure on $h\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure then we denote $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ as the full subcategories of $\mathcal{C}$ spanned by those objects belonging to $h\mathcal{C}_{\geq n}$ and $h\mathcal{C}_{\leq n}$ respectively.

Example 1. We state the main examples and in the course of this talk we shall verify the axioms in these cases.

- On the ∞-category of Sp we may define $Sp_{\geq 0}$ as the full subcategory spanned by spectra X such that $\pi_i X \simeq 0$ for $i < 0$ and $Sp_{\leq 0}$ as the full subcategory spanned by spectra Y such that $\pi_i Y \simeq 0$ for $i > 0$.
- Let R be a commutative ring and $\mathcal{D} := D^-(R)$ be the left bounded derived category of R. Then we may define $\mathcal{D}_{\geq 0}$ as the full subcategory spanned by the complexes X such that $H_i(X) \simeq 0$ for $i < 0$ and $\mathcal{D}_{\leq 0}$ as the full subcategory spanned by complexes X such that $H_i(X) \simeq 0$ for $i > 0$.

Now we will state a fact from Higher Topos Theory that we will use often.

Fact 2. HTT 5.2.7.8 Let $\mathcal{C}$ be an ∞-category and $\mathcal{C}^0 \subseteq \mathcal{C}$ a full subcategory. The following are equivalent:

- For every object there exists a localization $f : X \to Y$ relative to $\mathcal{C}^0$. That is for each $X \in \mathcal{C}$ there exists $Y \in \mathcal{C}^0$ and a map $X \xrightarrow{f} Y$ inducing an equivalence
$$\mathrm{Map}_{\mathcal{C}}(Y, W) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(X, W) \quad \forall W \in \mathcal{C}^0.$$
- The inclusion $\mathcal{C}^0 \subseteq \mathcal{C}$ admits a left adjoint.

Proposition 1. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, then for $n \in \mathbb{Z}$ the full subcategory $\mathcal{C}_{\leq n}$ is a localisation of $\mathcal{C}$.

Proof. Without loss of generality we assume $n = -1$, we will use **Fact 2**. Consider $X \in \mathcal{C}$ and from $(t3)$ we obtain a fibre sequence

$$X' \to X \xrightarrow{f} X''$$

with $X'' \in \mathcal{C}_{\leq -1}$. We claim f is our desired localisation relative to $\mathcal{C}_{\leq -1}$. For any $Y \in \mathcal{C}_{\leq 1}$ we have

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is an equivalence by Whitehead's Theorem as the fibre is $\mathrm{Map}_{\mathcal{C}}(X'[1], Y) \simeq 0$ by $(t1)$. □

For $n \in \mathbb{Z}$ we have shown the existence of a left adjoint $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$ to the inclusion $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$. A dual argument shows the inclusion $\mathcal{C}_{\geq n} \hookrightarrow \mathcal{C}$ admits a right adjoint $\tau_{\geq n}$. Hence:

Corollary 1. $\mathcal{C}_{\leq n}$ is stable under all limits that exist in $\mathcal{C}$ and dually $\mathcal{C}_{\geq n}$ is stable under all colimits that exist in $\mathcal{C}$.

Proof. This is due to the inclusion functors being appropriate adjoints. □

It follows from the above proof that:

Corollary 2. Given $\mathcal{C}$ as above, $X \in \mathcal{C}$ and $n \in \mathbb{Z}$ then there is a fibre sequence

$$\tau_{\geq n} X \to X \to \tau_{\leq n-1} X.$$

Fact 3. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure and $m, n \in \mathbb{Z}$. Then:
(a) $\tau_{\leq n} : \mathcal{C}_{\leq m} \to \mathcal{C}_{\leq m}$,
(b) $\tau_{\geq n} : \mathcal{C}_{\leq m} \to \mathcal{C}_{\leq m}$

Lemma 1. Let $a \leq b \in \mathbb{Z}$ and suppose $\exists f : X \to Y$ such that $\pi_i f$ is an equivalence for $a < i < b$. Then $\tau_{[a,b]} f := (\tau_{\geq a} \circ \tau_{\leq b}) f$ is an equivalence.

Proof. Clearly $\tau_{[i,i]}$ is an equivalence for $a < i < b$. Note we have a fibre sequence

$$\tau_{\geq a+1} \tau_{[a,a+1]} X \simeq \tau_{[a+1,a+1]} X \to \tau_{[a,a+1]} X \to \tau_{[a,a]} X \simeq \tau_{\leq n} \tau_{[a,a+1]} X.$$

Then f induces a map of fibre sequences:

$$\begin{array}{ccccc} \Omega\tau_{[a,a]}X & \longrightarrow & \tau_{[a+1,a+1]}X & \longrightarrow & \tau_{[a,a+1]}X \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \tau_{[a,a+1]}f \\ \Omega\tau_{[a,a]}Y & \longrightarrow & \tau_{[a+1,a+1]}Y & \longrightarrow & \tau_{[a,a+1]}Y \end{array}$$

and we conclude $\tau_{[a,a+1]} f$ is an equivalence and the claim follows from iterating this argument. □

Proposition 2. Let $\mathcal{C}, (\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$ be as above, then for all $m, n \in \mathbb{Z}$ there exists a canonical map

$$\tau_{\leq m} \circ \tau_{\geq n} \xrightarrow{\theta} \tau_{\geq n} \circ \tau_{\leq m}$$

which is an equivalence of functors $\mathcal{C} \to \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$.[1]

Proof. The existence of θ follows from abstract facts in HTT.7.3.1 and the commutativity of

$$\begin{array}{ccc} \mathcal{C}_{\geq n} & \longrightarrow & \mathcal{C} \\ \tau_{\geq n} \downarrow & & \downarrow \tau_{\leq m} \\ \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} & \longrightarrow & \mathcal{C}_{\leq m} \end{array} .$$

For each $X \in \mathcal{C}$ we have a morphism $\theta_X : \tau_{\leq m} \circ \tau_{\geq n} X \to \tau_{\geq n} \circ \tau \leq mX$, we wish to show this is an isomorphism in the homotopy category. If $m \leq n$ then both sides are zero so we are done. Assume $m \geq n$, it suffices to show that composition with θ_X induces an isomorphism

$$\mathrm{Ext}^0_{\mathcal{C}}(\tau_{\geq n} \circ \tau_{\leq m} X, Y) \xrightarrow{\theta_X^*} \mathrm{Ext}^0_{\mathcal{C}}(\tau_{\leq m} \circ \tau_{\geq n} X, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(\tau_{\geq n} X, Y)$$

[1]By intersection we mean the fibered product of the two inclusions into $\mathcal{C}$.

for all $Y \in \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$ where the last equivalence is by adjunction. We obtain a map of long exact sequences:

$$\begin{array}{ccc}
\mathrm{Ext}^0_{\mathcal{C}}(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_0} & \mathrm{Ext}^0_{\mathcal{C}}(\tau_{\leq n-1}X, Y) \\
\downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} \\
\mathrm{Ext}^0_{\mathcal{C}}(\tau_{\leq m}X, Y) & \xrightarrow{f_1} & \mathrm{Ext}^0_{\mathcal{C}}(X, Y) \\
\downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon^*} \\
\mathrm{Ext}^0_{\mathcal{C}}(\tau_{\geq n}\tau_{\leq m}X, Y) & \xrightarrow{f_2} & \mathrm{Ext}^0_{\mathcal{C}}(\tau_{\geq n}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathcal{C}}(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_3} & \mathrm{Ext}^1_{\mathcal{C}}(\tau_{\leq n-1}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathcal{C}}(\tau_{\leq m}X, Y) & \xrightarrow{f_4} & \mathrm{Ext}^1_{\mathcal{C}}(X, Y)
\end{array} \quad .$$

Since $m \geq n$ we have $\tau_{\leq n-1} \simeq \tau_{\leq n-1}\tau_{\leq m}$ we have that f_0 and f_3 are bijective. Since $Y \in \mathcal{C}_{\leq m}$ we have that f_1 is bijective and f_4 is injective (by the long exact sequence on homotopy groups of the relevant mapping spectra). Hence by the 5-lemma f_2 is bijective as desired. □

Definition 4. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then the **heart** $\mathcal{C}^\heartsuit$ of $\mathcal{C}$ is the full subcategory $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$. We denote $\pi_0 := \tau_{\geq 0} \circ \tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^\heartsuit$ and for $n \in \mathbb{Z}$ we let $\pi_n : \mathcal{C} \to \mathcal{C}^\heartsuit$ be the composition of π_0 with the shift $X \to X[-n]$.

Proposition 3. Let $\mathcal{C}$ be as above, then $\mathcal{C}^\heartsuit$ is equivalent to the nerve[2] of a 1-category.

Proof. Let $X, Y \in \mathcal{C}^\heartsuit$ then we shall compute the higher homotopy groups of

$$\mathrm{Map}_{\mathcal{C}^\heartsuit}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, Y).$$

We observe for $i \geq 1$:

$$\pi_i \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \pi_0 \Omega^i \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y[-i]) \simeq 0,$$

and hence the claim follows. □

In the proof it was shown that for X, Y in the heart of $\mathcal{C}$ that $\mathrm{Map}(X, Y) \simeq \pi_0 \mathrm{Map}(X, Y)$ and since stable categories are enriched over spectra we note the heart is enriched over abelian groups.

Fact 4. The heart of a stable ∞-category $\mathcal{C}$ is an abelian category and the data of a fibre sequence $X \to Y \to Z$ in $\mathcal{C}$ yields a long exact sequence of homotopy objects

$$\cdots \to \pi_n X \to \pi_n Y \to \pi_n Z \to \pi_{n-1} X \to \ldots$$

in $\mathcal{C}^\heartsuit$.

Lemma 2. Let $a < b \in \mathbb{Z}$ and suppose $\exists f : X \to Y$ such that $\pi_i f$ is an equivalence for $a \leq i \leq b$. Then $\tau_{[a,b]} f := (\tau_{\geq a} \circ \tau_{\leq b}) f$ is an equivalence.

Proof. Clearly $\tau_{[i,i]}$ is an equivalence for $a < i < b$. Note we have a fibre sequence

$$\tau_{\geq a+1}\tau_{[a,a+1]}X \simeq \tau_{[a+1,a+1]}X \to \tau_{[a,a+1]}X \to \tau_{[a,a]}X \simeq \tau_{\leq a}\tau_{[a,a+1]}X.$$

[2] We will ignore nerves in our notation.

Then f induces a map of fibre sequences:

$$\begin{array}{ccccc} \Omega\tau_{[a,a]}X & \longrightarrow & \tau_{[a+1,a+1]}X & \longrightarrow & \tau_{[a,a+1]}X \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\tau_{[a,a+1]}f \\ \Omega\tau_{[a,a]}Y & \longrightarrow & \tau_{[a+1,a+1]}Y & \longrightarrow & \tau_{[a,a+1]}Y \end{array}$$

and we conclude $\tau_{[a,a+1]}f$ is an equivalence and the claim follows from iterating this argument. □

Definition 5. Let $\mathcal{C}$ be a presentable stable ∞-category, then a t-structure on $\mathcal{C}$ is **accessible** if $\mathcal{C}_{\geq 0}$ is presentable.

Fact 5. For $\mathcal{C}$ a presentable stable ∞-category with t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ the following are equivalent:

- The t-structure is accessible.
- The ∞-category $\mathcal{C}_{\geq 0}$ is accessible.
- The ∞-category $\mathcal{C}_{\leq 0}$ is accessible.
- The ∞-category $\mathcal{C}_{\leq 0}$ is presentable.
- The truncation functor $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.
- The truncation functor $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.

4. Completeness & Whitehead

Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure.

Definition 6. We let $\mathcal{C}^+ := \bigcup \mathcal{C}_{\leq n}$, $\mathcal{C}^- := \bigcup \mathcal{C}_{\geq -n}$ and define $\mathcal{C}^b := \mathcal{C}^- \cap \mathcal{C}^+$. Then we say $\mathcal{C}$ is **left bounded** if $\mathcal{C} \simeq \mathcal{C}^+$, $\mathcal{C}$ is **right bounded** if $\mathcal{C} \simeq \mathcal{C}^-$ and **bounded** if $\mathcal{C} \simeq \mathcal{C}^b$.

Recall: Given a commutative ring R and an ideal I we may define the completion of R at I as the inverse limit over n of R/I^n where the map $R/I^{n+1} \to R/I^n$ is given by killing I^n. In what follows one observes an analogy to completion of rings by thinking of $\tau_{\leq 0}$ as "killing I" and $\tau_{\leq n}$ as "killing I^n.

Definition 7. We define the **left completion of** $\mathcal{C}$, denoted $\widehat{\mathcal{C}}$, as the limit of the tower

$$\cdots \to \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \ldots$$

and we say $\mathcal{C}$ is **left complete** if the canonical map $\mathcal{C} \xrightarrow{\simeq} \widehat{\mathcal{C}}$ is an equivalence.

We now note that being left complete is equivalent to the assertion $X \xrightarrow{\simeq} \varprojlim \tau_{\leq n} X$. There is an analogous notion of right completeness which asserts $\varinjlim \tau_{\geq n} X \xrightarrow{\simeq} X$.

Using results from section 3.3.3 Higher Topos Theory the ∞-category $\widehat{\mathcal{C}}$ as the full subcategory of $\mathrm{Fun}(N(\mathbb{Z}), \mathcal{C})$ spanned by the functors F such that:

- For each $n \in \mathbb{Z}$, $F(n) \in \mathcal{C}_{-n}$.
- For each $m \leq n \in \mathbb{Z}$, the induced morphism $F(m) \to F(n)$ induces an equivalence $\tau_{\leq -n} F(m) \xrightarrow{\simeq} F(n)$.

Proposition 4. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then:

(a) The left completion $\widehat{\mathcal{C}}$ is also stable.
(b) Let $\widehat{\mathcal{C}}_{\leq 0}$ and $\widehat{\mathcal{C}}_{\geq 0}$ be the full subcategories of $\widehat{\mathcal{C}}$ spanned by those functors $F : N(\mathbb{Z}) \to \mathcal{C}$ which factor through $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$, respectively. Then these subcategories determine a t-structure on $\widehat{\mathcal{C}}$.
(c) There is a canonical exact functor $\mathcal{C} \xrightarrow{F} \widehat{\mathcal{C}}$ which induces an equivalence[3] $\mathcal{C}_{\leq 0} \xrightarrow{\simeq} \widehat{\mathcal{C}}_{\leq 0}$.

[3] This part bolsters the above analogy with classical completion of rings.

Proof. (a) Omitted as this makes use of general facts about spectrum objects which will lead us too far astray from out interests.

(b) We note for $F \in \widehat{\mathcal{C}}$ we may define $F[n](m) := F(m+n)[n]$, yielding that $\widehat{\mathcal{C}}_{\geq 0}[1] \subseteq \widehat{\mathcal{C}}_{\geq 0}$ and $\widehat{\mathcal{C}}_{\leq 0}[-1] \subseteq \widehat{\mathcal{C}}_{\leq 0}$. Moreover it's a general fact that mapping anima in limits are simply the limits of the mapping anima, so for $X \in \widehat{\mathcal{C}}_{\geq 0}$ and $Y \in \widehat{\mathcal{C}}_{\leq 0}$ we observe:

$$\mathrm{Map}_{\widehat{\mathcal{C}}}(X,Y) \simeq \varprojlim_n \mathrm{Map}_{\mathcal{C}}(X(n),Y(n)) \simeq \varprojlim_n 0 \simeq 0$$

by (t_1). Now we produce the required fibre sequences, let $X \in \widehat{\mathcal{C}}$. We let $X'' := \tau_{\leq -1} \circ X : N(\mathbb{Z}) \to \mathcal{C}$ and note the unit of this adjunction assembles into a map $X \to X''$ where $X'' \in \widehat{\mathcal{C}}_{\leq -1}$. Then the fibre of this map is in $\widehat{\mathcal{C}}_{\geq 0}$ (as it can be identified with $\tau_{\geq 0} \circ X$.

(c) Let $\mathcal{D}$ denote the full subcategory of $\mathbb{Z} \times \mathcal{C}$ spanned by pairs (n, C) where $n \in \mathcal{C}_{\leq -n}$. It follow from **Fact 2** that the inclusion $\mathcal{D} \subseteq \mathbb{Z} \times \mathcal{C}$ admits a left adjoint L. By the hom-product adjunction in Cat_∞ the composition

$$\mathbb{Z} \times \mathcal{C} \xrightarrow{L} \mathcal{D} \subseteq \mathbb{Z} \times \mathcal{C} \xrightarrow{\pi_2} C$$

can be identified with a functor $\mathcal{C} \xrightarrow{\theta} \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ which factors through $\widehat{\mathcal{C}}$. It suffices to show θ is right exact[4] (as both left and right exactness are equivalent to exactness). Since the truncation functors $\tau_{\leq n} : \mathcal{C}_{\leq n+1} \to \mathcal{C}_{\leq n}$ are right exact, finite colimits in $\widehat{\mathcal{C}}$ are computed pointwise. Hence we are reduced to proving that each compostion

$$\mathcal{C} \xrightarrow{\theta} \widehat{\mathcal{C}} \to \tau_{\leq n}\mathcal{C}$$

is right exact. But this composition can be identified with the functor $\tau_{\leq n}$ (which is clearly right exact).

Finally, we observe that $\widehat{\mathcal{C}}_{\leq 0}$ can be identified with the limit of the essentially constant tower

$$\dots \xrightarrow{id} \mathcal{C}_{\leq 0} \xrightarrow{id} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \mathcal{C}_{\leq -1} \xrightarrow{\tau_{\leq -2}} \dots$$

and that θ induces an equivalence of this limit with $\mathcal{C}_{\leq 0}$.

□

Fact 6. There is an equivalence of ∞-categories between the category of left bounded stable ∞-categories and the ∞-category of left complete categories given on objects by $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ and $\mathcal{C} \mapsto \mathcal{C}^+$.

We give a criterion to check if a t-structure is left complete:

Proposition 5. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Suppose $\mathcal{C}$ admits countable products and that $\mathcal{C}_{\geq 0}$ is stable under them. Then $\mathcal{C}$ is left complete if and only if the full subcategory $\mathcal{C}_{\geq \infty} := \bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$ consists only of zero objects of $\mathcal{C}$.

Proof. We observe every tower

$$\dots X_n \to X_{n-1} \to \dots$$

in $\mathcal{C}$ admits a limit $\varprojlim X_n$ which belongs to $\mathcal{C}_{\leq -1}$. The functor $\mathcal{C} \xrightarrow{F} \widehat{\mathcal{C}}$ constructed above admits a left adjoint G given on objects by $\widehat{\mathcal{C}} \ni f \mapsto \varprojlim f$. Then $\mathcal{C} \xrightarrow{\simeq} \widehat{\mathcal{C}}$ is equivalent to the unit and counit maps:

$$u : F \circ G \to 1_{\widehat{\mathcal{C}}}$$
$$v : 1_C \to G \circ F$$

being equivalences. If v is an equivalence then X can be recovered as the limit of the tower $\{\tau_{\leq n} X\}$ and this implies the forward direction.

[4]Recall right exact means the functor commutes with finite colimits.

Now we assume $\mathcal{C}_{\geq\infty}$ consists only of zero objects. To prove u is an equivalence we must show for $f \in \widehat{\mathcal{C}}$ the projection map $\varprojlim(f) \xrightarrow{p_n} f(n)$ induces an equivalence $\tau_{\leq -n} \varprojlim(f) \to f(n)$. Equivalently we may show that the fibre of p_n is in $\mathcal{C}_{\geq -n+1}$. By the definition of a limit p_n factors as

$$\varprojlim(f) \xrightarrow{p_{n-1}} f(n-1) \xrightarrow{s_n} f(n).$$

By the octahedral axiom we obtain a fibre sequence

$$\mathrm{fib}(p_{n-1}) \to \mathrm{fib}(p_n) \to \mathrm{fib}(s_n).$$

Then since $\mathrm{fib}(s_n)$ is in $\mathcal{C}_{\geq -n+1}$ it suffices to show that $\mathrm{fib}(p_{n-1})$ is in $\mathcal{C}_{\geq -n+1}$ (we will see this is closed under extensions). We observe that the fibre of p_{n-1} can be identified with the limit of a tower $\{\mathrm{fib}(f(m) \to f(n-1))\}_{m \leq n}$ (limits commute with limits). Our claim follows from the fact that each $\mathrm{fib}(f(m) \to f(n-1)) \in \mathcal{C}_{\geq -n+2}$.
Now we consider v: let $X \in \mathcal{C}$ and consider $v_X : X \to (G \circ F)(X)$. Since u is an equivalence we conclude that $\tau_{\leq n}(v_X)$ is an equivalence for all $n \in \mathbb{Z}$. It follows that $\mathrm{cofib}(v_X) \in \mathcal{C}_{\geq n+1}$ for all $n \in \mathbb{Z}$. Then by assumption we conclude $\mathrm{cofib}(v_X) \simeq 0$ so that v_X is an equivalence as desired. □

This result will imply that Sp and $\mathcal{D}^-(R)$ for a commutative ring R are left and right complete.

Theorem 3 (Whitehead's Theorem)**.** Let C be a stable ∞-category equipped with a t-structure which is both left and right complete. Suppose $X \xrightarrow{f} Y$ is such that $\pi_i f$ is an isomorphism for all i, then f is an equivalence.

Proof. Using **Lemma 2**, for all $a, b \in \mathbb{Z}$ we have $\tau_{[a,b]} f$ is an equivalence. Since $f = \varprojlim_n \tau_{\leq n} f$ and $f = \varinjlim_n \tau_{\geq n} f$ we conclude f is an equivalence. □

5. t-structure on the left bounded derived category of a ring

We will construct a t-structure on the left bounded[5] derived category of a commutative ring R. Since the data of a t-structure is one-categorical in nature we will permit ourselves to work with the classical one-categorical left bounded derived category, that is the localisation of $Ch^-(R)$ at all quasi-isomorphisms. We consider $\mathcal{D} = D^-(R)$ and define $\mathcal{D}_{\geq 0}$ as the full subcategory of $\mathcal{D}$ spanned by complexes X such that $H_i(X) \simeq 0$ for $i < 0$ and similarly define $\mathcal{D}_{\leq 0}$ as the full subcategory spanned by complexes X such that $H_i(X) \simeq 0$ for $i > 0$. We will recall some facts which will be used to verify the axioms:

- For $X = (X_n), Y = (Y_n) \in Ch(R)$ we have
$$\mathrm{Hom}_{Ch(R)}(X, Y) := \{\mathrm{Hom}_{Ch(R)}(X, Y)_p\}_{p \in \mathbb{Z}}$$
where $\mathrm{Hom}_{Ch(R)}(X, Y)_p := \prod_n \mathrm{Hom}_R(X_n, Y_{n+p})$.
- The inclusion $\mathcal{D}_{\leq n} \hookrightarrow \mathcal{D}$ admits a left adjoint $\tau_{\leq n}$ (this can be proved by hand without general t-structure machinery).
- A triangle $X \to Y \to Z$ in $\mathcal{D}$ gives rise to a long exact sequence on homology groups
$$\cdots \to H_n X \to H_n Y \to H_n Z \to H_{n-1} X \to \dots$$
in the category of R modules Mod_R (again this is proved by hand using the snake lemma and that the localisation functor preserves homology groups).

Proposition 6. Let R be a commutative ring, $\mathcal{D} = D^-(R)$ and $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ defined as above. Then this is the data of a t-structure on $\mathcal{D}$.

[5] Recall $Ch^-(R)$ is the full subcategory of $Ch(R)$ spanned my complexes $M = (M_n)$ such that $M_n \cong 0$ for $n \ll 0$.

Proof. We check the axioms.

(a) Let $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq 0}$, we must show $\mathrm{Hom}(X,Y) \cong 0$. Choose X' and Y' lifts of X, Y respectively in $Ch^{-}(R)$. Since we wish to prove a statement in $\mathcal{D}$ we may choose X' such that $X'_i = 0$ for $i < 0$ and each X'_i is projective and Y' such that $Y'_j = 0$ for $j > -1$ and each Y'_j is injective. Then clearly $\mathrm{Hom}_{Ch(R)}(X', Y') = 0$, since X' and Y' are not homotopy equivalent (unless they are both 0, in which case the claim still follows) we obtain that $0 \cong \mathrm{Hom}_{Ch(R)}(X', Y') \cong \mathrm{Hom}_{K^{-}(R)}(X', Y') \cong \mathrm{Hom}(X, Y)$.
(b) Clearly $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$.
(c) Consider $X \in \mathcal{D}$ then by adjunction we obtain a map $X \to \tau_{\leq -1}X$ and by checking on the long exact sequence on homology groups we observe the fibre of this map is in $\mathcal{D}_{\geq 0}$.

□

We conclude with the following:

Proposition 7. Let $\mathcal{D} = D^{-}(R)$ and $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ be as above, then $\mathcal{D}^{\heartsuit} \xrightarrow[\simeq]{H_0} Mod_R$.

Proof. It is clear from the definitions that $\mathcal{D}^{\heartsuit}$ is the full subcategory of $\mathcal{D}$ spanned by complexes X such that $H_i(X) \simeq \begin{cases} M \in Mod_R & i = 0 \\ 0 & \text{else} \end{cases}$. Hence clearly H_0 is essentially surjective. We also have a functor $Mod_R \xrightarrow{i} \mathcal{D}$ given by sending a module M to the chain complex $\tilde{M}$ which has M in degree 0 and 0 everywhere else. Clearly i factors through $\mathcal{D}^{\heartsuit}$ and one can check $i \circ H_0 \simeq 1_{Ab}$ and $H_0 \circ i \simeq 1_{\mathcal{D}^{\heartsuit}}$.

□

6. t-structure on Sp

In contrast to the previous section we will now develop the necessary machinery to construct t-structures on "the category of spectrum objects in a presentable stable ∞-category $\mathcal{C}$" of which $Sp := Sp(An_*)$ is a special case.
In light of **Proposition** 1 we observe that t-structures give rise to localisations, in fact they correspond to a particular class of localisation.

Recall: If a full subcategory $\mathcal{C}'$ of a stable ∞-category $\mathcal{C}$ is *stable under extensions* if for all fiber sequences

$$X \to Y \to Z$$

in $\mathcal{C}$ it holds that

$$X, Z \in \mathcal{C}' \implies Y \in \mathcal{C}'.$$

The following proposition will be used to produce a t-structure on spectra.

Proposition 8. Let $\mathcal{C}$ be a stable ∞-category, let $L : \mathcal{C} \to \mathcal{C}$ be a localisation functor. Then the following are equivalent:

(a) The essential image of L is closed under extensions.
(b) The full subcategories $\mathcal{C}_{\geq 0} := \{A : LA \simeq 0\}$ and $\mathcal{C}_{\leq -1} := \{A : LA \simeq A\}$ determines a t-structure on $\mathcal{C}$.

Proof. We will only show $a) \implies b)$ as this is what we will make use of. Suppose the essential image of L is closed under extensions. For $A \in \mathcal{C}$ and $B \in L\mathcal{C}$ we will show the natural map $\mathrm{Ext}^1_{\mathcal{C}}(LA, B) \to \mathrm{Ext}^1_{\mathcal{C}}(A, B)$ is injective. We consider $\phi \in \mathrm{Ext}^1_{\mathcal{C}}(LA, B)$ and note by iteratively taking fibres we may produce a triangle

$$B \to C \xrightarrow{g} LA \xrightarrow{\phi} B[1].$$

Since B and LA are in $L\mathcal{C}$ and the image of L is closed under extensions we conclude $C \in L\mathcal{C}$. Suppose the image of ϕ in $\mathrm{Ext}^1_{\mathcal{C}}(A, B)$ is trivial, then by the universal property of the fibre the unit map $A \to LA$ factors as

$$A \xrightarrow{f} C \xrightarrow{g} LA.$$

Now we apply L to this diagram and since L is already local we observe that g admits a section and thus $\phi = 0$ (yielding the desired injectivity).
Now we will verify the axioms for our candidate t-structure:

(a) If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, then

$$\mathrm{Ext}^0_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(LX, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(0, Y) \simeq 0.$$

(b) Since $\mathcal{C}_{\leq -1}$ is a localisation of $\mathcal{C}$ it is stable under limits, hence $\mathcal{C}_{\leq -1}[-1] \subseteq \mathcal{C}_{\leq -1}$. Furthermore since L preserves colimits $LX \simeq 0 \implies L(X[1]) \simeq 0$ and hence $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$.

(c) For $X \in \mathcal{C}$, by taking the fibre of the unit map $X \to LX$ we produce a fibre sequence $X' \to X \to LX$ and now we claim $X' \in \mathcal{C}_{\geq 0}$. It suffices to show for all $Y \in L\mathcal{C}$ that $\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) = 0$. Since Y is local we have isomorphisms

$$\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(X', Y) \simeq \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y)$$

where the last equivalence comes from the definition of $\mathrm{Ext}^i_{\mathcal{C}}$. We now consider the exact sequence

$$\mathrm{Ext}^0_{\mathcal{C}}(LX, Y) \xrightarrow{f} \mathrm{Ext}^0_{\mathcal{C}}(X, Y) \xrightarrow{0} \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y) \to \mathrm{Ext}^1_{\mathcal{C}}(LX, Y) \xhookrightarrow{f'} \mathrm{Ext}^1_{\mathcal{C}}(X, Y)$$

where f is bijective since Y is local (equivalently by adjunction) and f' is injective by assumption, hence

$$0 \simeq \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(LX', Y)$$

and we are done.

□

Proposition 9. Let $\mathcal{C}$ be a presentable stable ∞-category. Suppose there exists a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ which is presentable, closed under small colimits and closed under extensions, then there exists[6] a t-structure on $\mathcal{C}$ such that $\mathcal{C}' =: \mathcal{C}_{\geq 0}$.

Proof. We will state some results from HTT to bootstrap the proof and then conclude by **Proposition 8**. We fix $X \in \mathcal{C}$ and let $\mathcal{C}'_{/X}$ denote the fibered product $\mathcal{C}_{/X} \times_{\mathcal{C}} \mathcal{C}'$. By HTT.5.5.3.12 $\mathcal{C}'_{/X}$ is presentable and in particular has a final object $Y \xrightarrow{f} X$. It follows that composition with f induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(Z, Y) \to \mathrm{Map}_{C}(Z, X) \quad \forall Z \in \mathcal{C}'.$$

Then by dualising **Fact 2** we observe $\mathcal{C}'$ is a colocalisation of $\mathcal{C}$. Since $\mathcal{C}'$ is stable under extensions the dual of **Proposition 8** gives the existence of a uniquely determined t-structure on $\mathcal{C}$ such that $\mathcal{C}' =: \mathcal{C}_{\geq 0}$. □

Proposition 10. Let $\mathcal{C} = Sp$ be the ∞-category of spectra and let $\mathcal{C}_{\leq -1}$ be the full subcategory spanned by objects X such that $\Omega^\infty X \simeq *$. Then $\mathcal{C}_{\leq -1}$ determines[7] an accessible t-structure on Sp.

[6]This t-structure is by definition accessible.

[7]Recall **Remark 1**.

Proof. We recall the functor $\Omega^\infty : Sp \to An$ admits a left adjoint Σ^∞_+. We recall the set $\{S^n\}_{n \in \mathbb{N} \cup \{0\}}$ generates An under colimits. Then we observe $E \in Sp_{\leq -1}$ if and only if for each n the spaces:

$$\mathrm{Map}_{An}(S^n, \Omega^\infty E) \simeq \mathrm{Map}_{Sp}(\Sigma^\infty_+ S^n, X)$$

are contractible. Then let $Sp_{\geq 0}$ be the smallest full subcategory of Sp which is stable under colimits, extensions and contains $\Sigma^\infty_+ S^n$. Then by **Proposition 9** $Sp_{\geq 0}$ is the data of an accessible t-structure on Sp. □

Now we compute its heart:

Proposition 11. There is an equivalence of categories $Sp^\heartsuit \xrightarrow[\simeq]{\pi_0} Ab$ with the category of abelian groups.

Proof. A spectrum E is in $Sp^\heartsuit$ if and only if it is an Eilenberg-MacLane spectrum $HA = \{K(A,n)\}_{n \in \mathbb{N}}$. We recall that

$$\mathrm{Hom}_{Ab}(A, B) \cong \mathrm{Map}_{An_*}(K(A,n), K(B,n))$$

for all $n \in \mathbb{N}$ and hence:

$$\mathrm{Map}_{Sp}(HA, HB) \simeq \lim_n \mathrm{Map}_{An_*}(K(A,n), K(B,n)) \simeq \mathrm{Hom}_{Ab}(A, B).$$

We conclude the functor π_0 is essentially surjective as well as fully faithful and hence an equivalence. □