Hermitian Symmetric Domains

Tejas Ramesh

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These are the notes I made for the talk I gave on Hermitian symmetric domains and their classification in terms of real algebraic groups, on 05-05-20205 at the MPI, as part of the student seminar *Shimura Varieties and their Canonical Models* organized by Dave (David Bowman) and Fabi (Fabian Schnelle). I follow Milne's notes *Introduction to Shimura Varieties*, chapter 01. Some material about differential geometry has been taken from Helgason's *Differential Geometry, Lie Groups and Symmetric Spaces*.

Motivation. As was dicussed in Talk 0, the upper half space H_1 is interesting in that it is "the mother of all $y(N)$", where $y(N)$ is the modular curve of level N. So, we want to understand the type of a geoemtric object H_1 is, which turns out to be that of hermitian symmetric domains. In this talk, we focussed on defining them and classifying them using real algebraic groups, which roughly mirrors connected Shimura data in its details.

1 A Quick Tour of Riemannian Geometry

1.1 Smooth Vector Fields and Smooth Tensor Fields

Recall. A topological manifold M is locally Euclidean, Hausdorff and second-countable. A smooth structure on M is a sheaf $\mathcal{O}_M$ of $\mathbb{R}$-valued functions st $(M, \mathcal{O}_M)$ is locally isomorphic to the $\mathbb{R}^n$ with its sheaf of $\mathbb{R}$-valued smooth functions. A smooth manifold is a topological manifold M with a smooth structure $\mathcal{O}_M$.
Let M be a smooth manifold. A tangent vector to M at p is an $\mathbb{R}$-derivation $\mathcal{O}_{M,p} \to \mathbb{R}$. The set of all tangent vectors at p to M is the $\mathbb{R}$-vector space T_pM, a basis in local co-ordinates $(x_1, \ldots, x_n)$ is given by,

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p,$$

and the dual basis is denoted $dx^1(p), \ldots, dx^n(p)$.
A continuous map between smooth manifolds is a smooth map if it is a morphism of ringed spaces, ie $f \in \mathcal{O}_N(U)$ for some $U \subseteq N$ open, implies $f \circ \alpha \in \mathcal{O}_M(\alpha^{-1}(U))$.
Given a smooth map $\alpha : M \to N$, the differential of α at $p \in M$ is,

$$d\alpha_p : T_pM \to T_{\alpha(p)}N,$$

$$X_p \mapsto X_p \circ \alpha^*.$$

A smooth vector field is a smooth section of the tangent bundle $\pi : TM \to M$, $(p, v) \mapsto p$. The set of all smooth vector fields on M is denoted $D^1(M)$ and it is a $\mathbb{R}$-vector space and a $C^\infty(M)$-manifold. That is, it is a choice of a tangent vector X_p at each $p \in M$ that varies smoothly over $p \in M$. An r-tensor field is a choice of r-mulilinear map $t_p : T_pM \times \ldots \mathbb{T}_pM \to \mathbb{R}$ for each $p \in M$, varying smoothly over $p \in M$.
Note: we can define an (r, s)-tensor field in the same way. For example, a $(1, 1)$-tensor field is a multi-linear map $T_pM \times T_pM^* \to \mathbb{R}$ for each $p \in M$ st this choice varies smoothly over $p \in M$.

Remark 1. A smooth $(1, 1)$-tensor field is the same thing as a family of endomorphisms $t_p : T_pM \to T_pM$ where the choice of t_p for each $p \in M$ varies smoothly over $p \in M$.

1.2 Riemannian Metric, Connections, Parallel Transport and Geodesics

Definition 1.1. A Riemannian metric is a smooth 2-tensor field g st,

$$g_p : T_pM \times T_pM \to \mathbb{R},$$

is symmetric and positive-definite for all $p \in M$. That is, a family of inner products g_p st the choice of g_p is smooth over $p \in M$. A Riemannian manifold is a smooth manifold with a Riemannian metric. An isometry is a diffeomorphism between Riemannian manifolds that preserves the metrics, ie $\varphi : (M, g) \to (N, h)$, st,

$$g_p(u, v) = h_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We write the group of isometries of (M, g) as Is(M,g).

Definition 1.2. A Riemannian or Levi-Civita connection ∇ on a Riemannian manifold (M, g) assigns to each $X \in D^1(M)$ an $\mathbb{R}$-linear map,

$$\nabla_X : D^1(M) \to D^1(M),$$

st,

1. $\nabla_{fX+Y} = f\nabla_X + \nabla_Y$,
2. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,
3. $X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
4. $\nabla_X Y - \nabla_Y X = [X, Y]$,

FACT 1. There is a unique Riemannian connection on any given Riemannian manifold.

Definition 1.3. Given $\gamma : [a, b] \to M$ is a smooth curve, $v \in T_pM$ with $p = \gamma(a)$; a parallel transport of v along γ is a vector field X on M st,

1. $X_p = v$,
2. $\nabla_{\gamma'(t)} X = 0$ for all $t \in [a, b]$,

Given a Riemannian manifold (M, g) with its unique connection ∇, we define its corresponding curvature tensor R to be a map,

$$R : D^1M^3 \to D^1(M),$$

$$(X, Y, Z) \mapsto R(X, Y)Z := ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(Z).$$

Let $\gamma : I \to M$ be a smooth curve in M; we say γ is a geodesic if $\nabla_{\gamma'(t)}\gamma'(t) = 0$

FACT 2. For any connected Riemannian manifold (M, g) and $v \in T_pM$, $\exists!$ maximal geodesic $\gamma : I \to M$ st $\gamma(0) = p$ and $\gamma'(0) = v$. By a maximal geodesic we mean that it is a geodesic that is NOT a proper restriction of any other geodesic.

Remark 2. Parallel transpost of tangent vectors along a geodesic: a vector in the tangent space is transported along a geodesic as the unique vector field with constant length and making a constant angle with the velocity vector of the geodesic.

2 Hermitian Symmetric Spaces

Remark 3. A complex vector space is precisely a real vector space V with an $\mathbb{R}$-linear endomorphism $J : V \to V$ st $J^2 = -id_V$, you need only specify how to do $i.v$, which is done by $J(v)$.

Definition 2.1. a Hermitian form on a complex vector space (V, J) is,

$$(.,.) : V \times V \to \mathbf{C},$$

st,

1. $(.,.)$ is a $\mathbb{R}$-bi-linear map,
2. $(Ju, v) = i(u, v)$,
3. $(v, u) = \overline{(u, v)}$

Recall. Let $f : \mathbf{C}^n \supset U \to \mathbf{C}$. We say, f is analytic if it admits a power series expansion in a neighborhood of each point of U. We say f is holomorphic if it is holomorphic ie complex differentiable separately in each variable. Hence, f is holomorphic iff it is analytic as in the case $n = 1$.
A complex manifold is a topological manifold M with a sheaf $\mathcal{O}_M$ of $\mathbf{C}$-valued functions, st $(M, \mathcal{O}_M)$ is locally isomorphic to $\mathbf{C}^n$ with its sheaf of analytic functions. A continuous function between complex manifolds is analytic if it is a morphism of ringed spaces.
A tangent vector at a point p of a complex manifold M is a $\mathbf{C}$-derivation $\mathcal{O}_{M,p} \to \mathbf{C}$. Locally, a basis is given by,

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}.$$

We denote by M^∞ the underlying $2n$-dimensional smooth manifold of an n-dimensional complex manifold. Then,

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n},$$

where,

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i},\right).$$

FACT 3. A smooth function $\alpha : M \to N$ (for M, N complex manifolds) is analytic $\iff$ $d\alpha_p$ is $\mathbf{C}$-linear for all $p \in M$.

Definition 2.2. An almost complex structure on a smooth manifold M is a smooth $(1, 1)$-tensor field $J = (J_p)_{p \in M}$ st $J_p^2 = -1$ for all $p \in M$. As noted above we think of the J_p as $\mathbb{R}$-linear endomorphisms of T_pM st $p \mapsto J_p$ is smooth.
Either ways, it is just a smoothly varying family of $\mathbf{C}$-linear structures on the tangent spaces of M.

Remark 4. Every complex manifold is an almost complex manifold, we take J_p to all be multiplication by i.

Definition 2.3. Let M be an (almost) complex manifold. Then, a Heremitian metric on M is a Riemannian metric g st,

$$g(JX, JY) = g(X, Y),$$

for all $X, Y \in D^1(M)$.

Remark 5. For each $p \in M$, g_p is the real part of a unique hermitian form h_p on T_pM.

Definition 2.4. A Hermitian manifold is a complex manifold M with a Hermitian metric g. The automorphism group of a Hermitian manifold is denoted $\text{Is}(M, g)$ and consists of all holomorphic isometries on (M, g). The group of automorphisms of a complex manifold M is denoted $\text{Hol}(M)$. Note,

$$\text{Is}(M, g) = \text{Is}(M^\infty, g) \cap \text{Hol}(M).$$

A Hermitian manifold (M, g) is homogeneous if $\text{Is}(M, g)$ acts transitively on M. It is symmetric if it is homogeneous and there is a symmetry s_p at some p in M, ie s_p is a automorphism of the manifold st $s_p^2 = 1$ and p is its only fixed point in some neighborhood of p. A connected symmetric Hermitian manifold is called a Hermitian symmetric space.

Example. (Upper Half Plane) Consider the complex upper half plane $H_1 := \{z \in \mathbf{C} \ : \ Im(z) > 0\}$ with a metric g given by,

$$g_p(x_1 + iy_1, x_2 + iy_2) := \frac{x_1x_2 + y_1y_2}{y^2}.$$

It is a complex manifold as it is open in $\mathbf{C}$. One can check that this g is indeed a Riemannian metric that is invariant under J given by multiplication by i. Hence (H_1, g) is a Heremitian manifold. Note that a Möbius transformation on $\mathbf{C}$ is any,

$$\mathbf{C} \to \mathbf{C} : z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$. Möbius transformations are conformal bijections on the extended complex plane with inverses which are also Möbius transformations. They are diffeormophsims since they are rational. The set of Möbius transformations that send H_1 to H_1 are precisely those where $ad - bc = 1$, ie elements of $SL_2(\mathbb{R})$. These Möbius transformations preserve g (I think this is rather finicky to check). So, $SL_2(\mathbb{R})/\{\pm I\} =: PSL_2(\mathbb{R}) \subseteq \text{Is}(H_1, g)$. (I suppose?) one can show that this is infact an equality since the elements of $\text{Is}(H_1, g)$ must preserve the metric g. Finally, $PSL_2(\mathbb{R})$ acts on H_1 by Möbius transformations, ie,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := \frac{az + b}{cz + d}.$$

This action is transitive since for any $z = x + iy \in H_1$, (note $y > 0$), you can go from i to z by,

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} .i = z.$$

Symmetry at i: consider $s_i : z \mapsto \frac{-1}{z}$, then,

$$i \mapsto \frac{-1}{i} = i,$$

$$z \mapsto \frac{-1}{z} \mapsto \frac{-1}{-1/z} = z.$$

The only other point it fixes in $\mathbf{C}$ is $-i$, but that is NOT in H_1.

Example. (Riemann Sphere) Consider the complex manifold $P^1(\mathbf{C})$ endowed with the restriction to the spehere of the standard metric $g = dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$. This is a Hermitian manifold. Rotations are holomorphic isometries and the group of rotations acts transitively on $P^1(\mathbf{C})$; a symmetry at the north pole is given by rotation by π about the axis connecting the north and the south poles.

Example. Consider $\mathbf{C}/\Lambda \cong \mathbf{C}/\mathbb{Z}$ (where Λ is some lattice in $\mathbf{C}$) endowed with the standard metric of $\mathbf{C}$. Tranlations act transitively on $\mathbf{C}/\Lambda$ and a symmetry at 0 is given by $z \mapsto -z$.

3 Sectional Curvature

Motivation. Sectional curvature is a well-defined intrinsic property of any Riemannian manifold (M, g). We classify Hermitian symmetric spaces into compact, non-compact and euclidean types, depending on whether they have positive curvature, negative curvature or zero curvature respectively.

Recall. (Gauss Curvature of a Surface) The curvature at a point of a curve is the radius of the best fitting circle to the curve at that point. For any point p on a surface, take a normal at p, and consider planes containing the normal. The planes intersect the surface in a curve. Take maximum $K_{\max,p}$ and minimum $K_{\min,p}$ of the signed curvatures of the curves at p, where the sign is positive if the curve bends towards the normal and negative otherwise. The Gaussian curvature $K(p)$ of the surface at p is the product $K_{\max,p}.K_{\min,p}$. We say that the surface has positive curvature if $K(p) > 0$, negative curvature if $K(p) < 0$ and zero curvature if $K(p) = 0$.

FACT 4. *(Gauss' Theorema Egregium) Gauss curvature is a well-defined intrinsic property of any 2-dimensional Riemannian manifold.*

Definition 3.1. Let (M, g) be a Riemannian manifold and $p \in M$. Consider E a 2-dimensional subspace of T_pM and the geodesics in M through p tangent to E. The sectional curvature $K(p, E)$ of M at p wrt E is the Gauss curvature of the 2-dimensional sub-manifold of M cut out by these geodesics.

Example. Intuitively, negative (resp. positive) curvature means, at each point, the geodesics diverge (resp. converge).

1. H_1 has negative curvature,
2. $P^1(\mathbf{C})$ has positive curvature,
3. $\mathbf{C}/\Lambda$ has zero curvature.

4 Hermitian Symmetric Domains

There are threee families of Hermitian symmetric spaces, they are,

1. non-compact type: ones with negative curvature,
2. compact type: positive curvature,
3. Euclidean type: zero curvature.

FACT 5. Every Hermitian symmetric space decomposes into a product $M^0 \times M^- \times M^+$ with M^0 Euclidean, M^- non-compact type and M^+ compact type,

Definition 4.1. A Hermitian symmetric domain is a hermitian symmetric space of non-compact type.

Example. (Siegel Upper Half Space) We generalize the Hermitian symmetric domain H_1. The Siegel upper half space of degree g denoted H_g is the set of all symmetric $g \times g$ matrices $Z = X + iY$ with complex entries st Y is positive definite (ie $x^T Y x > 0$ for all $x \neq 0$). The map,

$$H_g \to \mathbf{C}^{g(g+1)/2},$$

$$(z_{i,j}) \mapsto (z_{i,j})_{j \geq i},$$

identifies H_g with an open subset of $\mathbf{C}^{g(g+1)/2}$, and is hence a complex manifold. The symplectic group,

$$Sp_{2g}(\mathbb{R}) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \; : \; A^T C = C^T A, B^T D = D^T B, D^T A - B^T C = I_g, A^T D - C^T B = I_g \right\},$$

is the group of holomorphic automorphisms of H_g. This group acts on H_g by Möbius, ie,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .Z := (AZ + B)(CZ + D)^{-1},$$

and this action is transitive(? I suppose it is similar to the $g = 1$ case) and a symmetry at iI_g is given by,

$$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \in Sp_{2g}(\mathbb{R}).$$

Hence H_g is a symmetric domain (ie a non-empty connected open subset of $\mathbf{C}^n$ for some n that has a symmetry at each point). We want to give H_g a Hermitian metric and make it a Hermitain symmetric domain. This is done by identifying H_g with D_g, noting that H_g and D_g are bounded symmetric domains and hence are Hermitian symmetric domains with the Bergman metric (see remark below).

Remark 6. 1. Define D_g to be the set of symmetric matrices Z with complex entries st $I_g - \overline{Z^T} Z$ is positive definite. The map $(z_{ij}) \mapsto (z_{ij})_{j \geq i})$ identifies D_g with a bounded domain in $\mathbf{C}^{g(g+1)/2}$. The map,

$$Z \mapsto (Z - iI_g).(Z + iI_g)^{-1},$$

is an isomorphism from H_g to D_g.

2. Every bounded domain has a canonical hermitian metric called the Bergman metric;

3. Aby bounded domain with its Bergman metric has negative curvature. Hence, every bounded symmetric domain is a hermitian symmetric domain with its Bergman metric.

5 Symmetric Spaces as Lie Groups Modulo a Compact Subgroup

FACT 6. Given a (hermitian) symmetric space (M, g), the group $\mathrm{Is}(M, g)$ of (holomorphic) isometries on (M, g) has a natural structure of a Lie group.

Theorem 5.1. Let (M, g) be a symmetric space and $p \in M$.
Let K_p be the subgroup of $\mathrm{Is}(M, g)^+$ that fixes p. Here, by $\mathrm{Is}(M, g)^+$ we mean the connected component of $\mathrm{Is}(M, g)$ containing the neutral element.
Then,

1. K_p is compact,

2. the map,
$$\mathrm{Is}(M, g)^+/K_p \to M,$$
$$a.K_p \mapsto a.p,$$
is an isomorphism of smooth manifolds,

3. $\mathrm{Is}(M, g)^+$ acts transitively on M.

Proof. 1. let (M, g) be a Riemannian manifold,

2. the compact open topology makes $\mathrm{Is}(M, g)$ into a locally compact group for which the stabilizer K'_p of some point $p \in M$ is compact,

3. the natural lie group structure on $\mathrm{Is}(M, g)$ is the unique lie group structure on $\mathrm{Is}(M, g)$ that is compatibe with the compact open topology,

4. $\mathrm{Is}(M, g)/K'_p \to M$ is a homeomorphism and the map,
$$\mathrm{Is}(M, g) \to M,$$
$$a \mapsto a.p,$$
is open,

5. writing,
$$\mathrm{Is}(M, g) = \sqcup_{i=1}^{l} \mathrm{Is}(M, g)^+.a_i,$$
for any two cosets, the open sets $\mathrm{Is}(M, g)^+.a_i.p$, $\mathrm{Is}(M, g)^+.a_j.p$ are either disjoint or equal,

6. but M is connected, so these open sets must be all equal,

7. hence $\mathrm{Is}(M, g)^+$ acts transitively on M,

8. since $\mathrm{Is}(M, g)/K'_p \to M$ is a homeomorphism and $\mathrm{Is}(M, g)^+$ acts transitively on M, we have that $\mathrm{Is}(M, g)^+/K_p \to M$ is a homeomorphism,

9. it is a diffeomorphism of smooth manifolds as it is a morphism of ringed spaces.

□

Proposition 5.1. Let (M, g) be a Hermitian symmetric domain. Then,

1. $\mathrm{Is}(M, g)^+ = \mathrm{Is}(M^\infty, g)^+ = \mathrm{Hol}(M)^+$,

2. $\mathrm{Hol}(M)^+$ acts transitively on M,

3. the stabilizer K_p of p in $\mathrm{Hol}(M)^+$ is compact,

4. $\mathrm{Hol}(M)^+/K_p \cong M^\infty$ in the category of smooth manifolds.

Proposition 5.2. Suppose (M, g) be a HSD and let $\mathfrak{h}$ denote the Lie algebra of $\mathrm{Hol}(M)^+$. Then, there is a unique connected algebraic subgroup G of $GL(\mathfrak{h})$ st $G(\mathbb{R})^+ \cong \mathrm{Hol}(M)^+$ (inside $GL(\mathfrak{h})$).

Remark 7. The previous proposition is rather unexpected!
Let A be a connected real Lie group with Lie algebra $\mathfrak{a}$. There need NOT, in general, be a real algebraic group G st $G(\mathbb{R})^+ = A$. See discussion right above Proposition 1.7 in Milne's notes for Introduction to Shimura Varieties.

6 The Homomorphism from the Circle Group

Let $U_1 := \{z \in \mathbf{C} \ : \ |z| = 1\}$, be the circle group. Inspired the connected Shimura datum and the previous proposition, we would like to consider homomorphisms $u_p : U_1 \to \mathrm{Hol}(D)$, for given Hermitian symmetric domain D.

Example. Let $p = i \in H_1$. Consider the homomorphism,

$$h : \mathbf{C}^\times \to SL_2(\mathbb{R}),$$

$$a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Note $h(z)$ acts on the tangent space $T_i(H_1)$ as multiplication by $\frac{a+ib}{a-ib}$, ie,

$$h(z).v := \frac{a + ib}{a - ib}.v,$$

for all $v \in T_i H_1$. For $z \in U_1$, choose a square root $\sqrt{z}$, and set $u(z) = h(\sqrt{z}) \mod \pm I$. Then,

$$u : U_1 \to PSL_2(\mathbb{R}),$$

$$z \mapsto h(\sqrt{z}),$$

is a well-defined map, since $u(z)$ does NOT depend on the choice of root of z as[1] $h(-1) = -I$. So, u is a homomorphism since h is. Moreover, $u(z)$ acts on $T_i H_1$ as multiplication by z, ie,

$$u(z).v := z.v,$$

for all $v \in T_i H_1$.

Theorem 6.1. Let D be a Hermitian symmetric domain. Then, $\forall\ p \in D$, $\exists!$ homomorphism,

$$u_p : U_1 \to \mathrm{Hol}(D),$$

st,

1. $u_p(z)$ fixes p,
2. $u_p(z)$ acts on $T_p M$ as multiplication by z.

Proof. 1. let (M, g) be a symmetric space and $p \in M$, then,

   (a) the symmetry s_p at p acts as -1 on $T_p M$ (ie $(ds_p)(X_p) = -X_p$) and $s_p \circ \gamma = (t \mapsto \gamma(-t))$, $s_p^2 = 1$ implies $(ds_p)^2 = 1$, so $ds_p(X_p) = \pm X_p$ work on basis vectors, show that if it is $+X_p$, then $s_p \circ \gamma$ is a geodesic containing the maximal geodesic γ (use reflection), so it has to be $-X_p$; the latter claim follows from the uniqueness of geodesics by showing that $t \mapsto \gamma(-t)$ and $s_p \circ \gamma$ have same initial point and initial velocity vector,

   (b) the pair (M, g) is geodesically closed[2], (if some maximal geodesic is not defined on all of $\mathbb{R}$, then $s_{\gamma(t_0)} \circ \gamma$ extends γ to a geodesic),

2. on a symmetric space (M, g), every canonical r-tensor[3] with r odd is zero, ($t_p = t_p \circ (ds_p)^r = (-1)^r t_p = -t_p$ since r is odd, hence $t_p = 0$),

3. on a symmetric space (M, g), parallel transport of two-dimensional subspaces does NOT change the sectional curvature, (for Riemannian connection ∇ and its corresponding curvature tensor R, we have $\nabla \circ R$ is a 3-tensor, and is hence 0),

4. the exponential map[4] $\exp_p$ is smooth on some open neighborhood of 0 in D_p,

5. if M is geodesically complete then $\exp_0$ is defined on the whole of $T_p M$,

6. let (M, g), (M', g') be Riemannian manifolds in wich parallel transport of 2-dimensional subspaces of tangent spaces does NOT change the sectional curvature, let $a : T_p M \to T_{p'} M'$ be a linear isometry st $K(p, E) = K(p', aE)$ for every 2-dimensional subspace $E \subsetneq T_p M$; then, $\exp_p(X) \mapsto \exp_{p'}(aX)$ is an isometry of a neighborhood of p onto a neighborhood of p',

[1] Here we use that there are exactly two square roots of any $z \in U_1$ and they differ by π in angle and that $e^{i\pi} = -1$.

[2] A Riemannian manifold is geodesically complete if every maximal geodesic is defined on the whole of $\mathbb{R}$.

[3] A canonical r-tensor on a symmetric space (M, g) is an r-tensor fixed by every isometry of (M, g) ie for each $p \in M$, t st $t_p \circ (d\sigma_p)^r = t_p$ for all $\sigma \in \mathrm{Is}(M, g)$.

[4] Let $v \in T_p M$, let $\gamma_v : I_v \to M$ denote the maximal geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, let D_p be the set of $v \in T_p M$ st I_v contains 1, then, there is a unique map $\exp_p : D_p \to M$, $tv \mapsto \gamma_v(t)$ whenever $tv \in D_p$.

7. additionally, if M, M' are geodesically complete, connected, and simply connected, then there is a unique isometry $\alpha : M \to M'$ st $\alpha(p) = p'$ and $(d\alpha)_p = a$ (the conditions imply that locally defined isometry extends globally),

8. let $p \in D$,

9. each $z \in U_1$ defines an automorphism of the vector space T_pD (via multiplication by z) preserving g_p and sectional curvatures,

10. by the above points, there is a unique isometry,

$$u_p(z) : D \to D,$$

fixing p and acting as multiplication by z on T_pD, ie $d(u_p)_p.v = z.v$,

11. $u_p(z)$ is holomorphic since it is smooth and $d(u_p)_p$ is a $\mathbf{C}$-linear map,

12. $u_p(z) \circ u_p(z')$ fixes p and acts as a multiplication by zz', hence by uniqueness it is equal to $u_p(zz')$,

13. so $u_p : U_1 \to \mathrm{Hol}(D)$ is a unique well-defined homomorphism st $u_p(z)$ fixes p and acts on T_pM by multiplication by z.

□

7 Representations of U(1)

Let T be a torus over a field k, we want to describe representations of T by finite dimensional k-vector spaces V.

Remark 8. (When T is a split torus) Suppose T is split. Then, every representation $\rho : T \to GL_V$ is diagonalizable. Hence,

$$V = \oplus_{\chi \in X^*(T)} V_\chi,$$

where V_χ is the subspace on which T acts through the character χ, ie,

$$\rho(t)v = \chi(t).v,$$

for $v \in V_\chi$, $t \in T(k)$.

Definition 7.1. When $V_\chi \neq 0$, we say χ occurs in V.

Remark 9. (When T splits only over a Galois extension K/k) Let V be a k-vector space and let ρ be a representation of T_K on $K \otimes_k V$. Then, as above,

$$K \otimes_k V = \oplus_{\chi \in X^*(T)} V_\chi.$$

FACT 7. To give a representation of T on a k-vector space V amounts to giving a gradation,

$$K \otimes_k V = \oplus_\chi V_\chi,$$

for which $\sigma V_\chi = V_{\sigma\chi}$ for all $\sigma \in Gal(K/k)$, $\chi \in X^*(T)$.

FACT 8. Consider U_1 as a real algebraic torus.

1. Its characters are all of the form $z \mapsto z^n$, $n \in \mathbb{Z}$,
2. $X^*(U_1) \cong \mathbb{Z}$, and complex conjugation is multiplication by -1,
3. any representation of U_1 on an $\mathbb{R}$-vector space is a gradation,

$$V \otimes_{\mathbb{R}} \mathbf{C} =: V(\mathbf{C}) = \oplus_{n \in \mathbb{Z}} V^n,$$

   st $V(\mathbf{C})^{-n} = \overline{V(\mathbf{C})^n}$ for all n, here V^n is $V_{\chi: z \mapsto z^n}$,

4. V^0 is defined over $\mathbb{R}$ since $V(\mathbf{C})^0 = \overline{V(\mathbf{C})^0}$.

Proposition 7.1. every real representation of U_1 is a direct sum of representations of the following types,

1. $V = \mathbb{R}$ with U_1 acting trivially (so $V(\mathbf{C}) = V^0$),
2. $V = \mathbb{R}^2$ with $x + iy \in U_1(\mathbb{R})$ acting as,

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^n,$$

   for $n > 0$ (so $V(\mathbf{C}) = V^n \oplus V^{-n}$),

these representations are irreducible and hence no two are isomorphic.

8 Classification of Hermitian Symmetric Domains in terms of Real Groups

Remark 10. Representations of U_1 have the same description whether we regard U_1 as a Lie group or as an algebraic group. Hence, every homomorphism $U_1 \to GL(V)$ of real Lie groups is a morphism of algebraic groups.

Proposition 8.1. Let D be an Hermitian symmetric domain. Then, the homomorphism,

$$u_p : U_1 \to \mathrm{Hol}(D)^+ \cong G(\mathbb{R})^+,$$

is a morphism of algebraic groups. See Proposition 5.2 and Theorem 6.1.

Recall. Given a Lie group G, we define the adjoint representation,

$$Ad : G \to \mathrm{Aut}(G),$$

$$g \mapsto Ad_g,$$

where $Ad_g = d(ad_g)_e : T_eG \to T_eG$, and $ad_g : G \to G$ is the inner automorphism by g.

Theorem 8.1. Let D be a HSD and let G be the associated real adjoint algebraic group (as in Proposition 5.2).
Then, the homomorphism $u_p : U_1 \to G$ attached to a point p of D satisfies,

1. only the characters $z, 1, z^{-1}$ occur in the representation of U_1 on $Lie(G)_{\mathbf{C}}$ defined by $Ad \circ u_p : U_1 \to G \to GL(Lie(G)_{\mathbf{C}})$, note this means $V_\chi \neq 0$ only for the characters associated to $n = -1, 0, 1$,
2. $\mathrm{Ad}(u_p(-1))$ is a Cartan involution,
3. $u_p(-1)$ does NOT project to 1 in any simple factor of G.

Conversely, let G be a real adjoint algebraic group and let $u : U_1 \to G$ satsify the conditions $1, 2, 3$ above. Then, the set D of conjugates of u by elements of $G(\mathbb{R})^+$ has a natural structure of a Hermitian symmetric domain for which $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ and $u(-1)$ is the symmetry at u (regarded as a point of D).

Corollary 8.1.1. There is a natural $1-1$-correspondence between the isomorphism classes of pointed Hermitian symmetric domains and pairs (G, u), where G is a real adjoint Lie group and $u : U_1 \to G(\mathbb{R})$ is a homomorphism satsifying the conditions $1, 2, 3$.

Example. Let $u : U_1 \to PSL_2(\mathbb{R})$ be as before (ie $u(z) = h(\sqrt{z}) \mod \pm I$). Then,

$$u(-1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

as one of the square roots of -1 is i. Note, $\theta := \mathrm{Ad}(u(-1))$ is a Cartan involution on SL_2, (since $SL_2^\theta(\mathbb{R})$ can identified with a closed bounded set in $\mathbf{C}^2$ and is hence compact), and hence also on PSL_2. Conditions 1 and 3 are trvially satisfied. Hence, the set D of conjugates of u by elements of $PSL_2(\mathbb{R})^+$ is a Hermitian symmetric domain for which $PSL_2(\mathbb{R}) = PSL_2(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ and $u(-1)$ is the symmetry at u. Furthermore, D is H_1 as $PSL_2(\mathbb{R}) \cong \mathrm{Is}(H_1)$.