Shimura varieties and their canonical models
Talk 2 — Miscellanea on algebraic and Lie groups

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1 Lie Groups and Algebraic Groups

1.1 Lie Groups

We start with defining Lie groups, and look at some examples thereof.

Definition 1.1 (Lie group)
A real (complex) ***Lie group*** *is a differentiable (complex) manifold with a group structure, such that multiplication and inversion are smooth (holomorphic) maps.*

In this seminar, we will focus mostly on real Lie groups, hence hereafter a Lie group is a real one, unless stated otherwise. We now list some examples of Lie groups.

Example 1.2

(1) *Every finite group is a Lie group, considering it as a discrete topological space and hence a zero-dimensional manifold.*

(2) *In dimension one, we have only two connected Lie groups: the real line $\mathbb{R}$ with addition as group operation, and S^1 considered inside $\mathbb{C}$ and with multiplication as the group operation.*

(3) *The general linear group $\mathrm{GL}_n(\mathbb{R})$ is a real Lie group of dimension n^2. The following subgroups of $\mathrm{GL}_n(\mathbb{R})$ are Lie groups, and are so-called* ***matrix Lie groups****.*
   - *The special linear group, $\mathrm{SL}_n(\mathbb{R})$, consisting of those invertible matrices with determinant 1, is a Lie group of dimension $n^2 - 1$.*
   - *The orthogonal group, $\mathrm{O}(n, \mathbb{R})$, consisting of those invertible matrices whose inverse is its tranpose (equivalently, they respect the standard dot product on $\mathbb{R}$), is a Lie group of dimension $n(n-1)/2$.*

- *The special orthogonal group,* $\mathrm{SO}(n,\mathbb{R})$*, the intersection on* $\mathrm{O}(n,\mathbb{R})$ *and* $\mathrm{SL}_n(\mathbb{R})$*, is also a Lie group of dimension* $n(n-1)/2$*.*
- *The symplectic group,* $\mathrm{Sp}(2n,\mathbb{R})$*, consisting of those matrices* M *of order* $2n \times 2n$ *satisfying* $M^T \Omega M = \Omega$ *is a Lie group of dimension* $n(2n+1)$*, where* $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$*. It is a subgroup of* SL_{2n}*.*

(4) *We similarly have the complex general linear group* $\mathrm{GL}_n(\mathbb{C})$*, with the Lie subgroups* $\mathrm{SL}_n(\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C})$*. Notice that the subgroups* $\mathrm{U}(n), \mathrm{SU}(n)$ *of orthogonal and special orthogonal matrices, respectively, is a subgroup, but only a real Lie group; their defining equations involve conjugation, which is not a holomorphism, so we do not have a complex structure on them.*

We now come to the most important example of Lie groups for the seminar, for which we need some preliminary definitions. A manifold is **homogeneous** if its automorphism group acts transitively on it. A manifold is **symmetric** if it is homogeneous and at some (equivalently by homogeneity, at each) point p there is a an involution s_p having p as a unique local fixed point, i.e. $s_p^2 = 1$ and p is the only fixed point of s_p in some neighbourhood of p. A connected symmetric Riemannian manifold is called a **symmetric space**. The group of isometries of a symmetric space has a canonical structure of a Lie group, with the group action given by composition.

Let M be a symmetric space, and denote by $I(M)$ its group of isometries. We will define the topology on $I(M)$ making it into a Lie group, though we will not prove that it is a Lie group. The interested reader may find it in [Hel25][§4, Lemma 3.2]. The topology on $I(M)$ is the so-called **compact open topology**. It is defined as follows: Let C, U be a compact and an open subset of M, respectively, and define

$$W(C,U) := \{g \in I(M) \mid g \cdot C \subseteq U\}.$$

We then let the topology on $I(M)$ be the one generated by these sets.

1.2 Algebraic Groups

We now define and list some examples of algebraic groups.

Definition 1.3 (Algebraic group)
An ***algebraic group*** *over a field* k *is an algebraic variety* G*, together with an element* $e \in G$ *(strictly,* $e \in G(k)$*), and regular maps* $m : G \times G \to G$ *and* $i : G \to G$*, such that they satisfy the usual group axioms, with* m *being multiplication,* i *inversion, and* e *the identity element.*

We now list some examples of algebraic groups.

Example 1.4

(1) *As we saw last lecture, by definition an abelian variety is an algebraic group.*

(2) *The affine line* $\mathbb{A}^1(k)$ *with the group structure of addition is an algebraic group. We denote it by* $\mathbb{G}_a(k)$*.*

(3) *Let* $\mathbb{G}_m(k)$ *be the affine variety defined by the equation* $xy = 1$ *in* $\mathbb{A}^1(k)$*. The pointwise multiplication and inversion maps are regular, with identity element* $(1,1)$*, and make* $\mathbb{G}_m(k)$ *into an algebraic group.*

(4) *The general linear group* $\mathrm{GL}_n(k)$ *is an algebraic group over* k *— it is cut out by the polynomial equation* $\det(x_{ij})t - 1 = 0$ *inside* $\mathbb{A}^{n^2+1}(k)$*, where we denote the last variable by* t*. Then, the its subgroups* $\mathrm{SL}_n(k), \mathrm{O}(n,k), \mathrm{SO}(n,k), \mathrm{Sp}(2n,k)$ *as above have definitions that are valid over any field, and make them into algebraic groups as well.*

We have the first equivalence theorem of a large class of algebraic groups.

Proposition 1.5
Let G *be an algebraic group over* k*. Then the underlying variety of* G *is affine, if and only if* G *is embeddable in some* $\mathrm{GL}_n(k)$*, i.e. there is a regular map* $G \hookrightarrow \mathrm{GL}_n(k)$*. We call such an algebraic group* ***affine*** *(after the former condition) or* ***linear*** *(after the latter).*

Notice that in particular, this implies that $\mathbb{G}_a(k)$ and $\mathbb{G}_m(k)$ are linear. Indeed, $\mathbb{G}_m(k) \simeq \mathrm{GL}_1(k)$, and one has the following faithful representation:

$$\begin{aligned} \mathbb{G}_a(k) &\hookrightarrow \mathrm{GL}_2(k) \\ a &\longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

1.3 Connection Between the two

We now explain some of the similiarities between Lie and algebraic groups.

Suppose G is an algebraic group over $\mathbb{R}$. If G is smooth[1], then G (or rather, $G(\mathbb{R})$) can be naturally given the structure of a smooth real manifold. As regular functions are smooth, this turns G into a Lie group.

So in some sense, algebraic groups generalise Lie groups to arbitrary fields. However, this is not strictly true, as some Lie groups do not arise from algebraic groups.

Moreover, in studying both (smooth) algebraic groups and Lie groups, one often uses their Lie algebras, to greater or lesser extent.

2 Interesting Algebraic Groups

2.1 Unipotent Groups

Let $U_n \subseteq \mathrm{GL}_n(k)$ denote the subgroup of all upper-triangular matrices with diagnoal 1.

We call a linear algebraic group G **unipotent** if it is isomorphic to a closed subgroup of U_n for some n. By a non-trivial theorem (see [Milc][§13]), this is equivalent to each element of G being unipotent, where $g \in G$ is called unipotent if for all representations $\rho : G \to \mathrm{GL}_n(k)$ we have $\rho(g)$ unipotent, i.e. is of the form $1 + n$ for some nilpotent $n \in \mathrm{GL}_n(k)$.

[1] We can define smoothness by the so-called Jacobian criterion, requiring the Jacobian matrix of the defining polynomials to be of maximal rank at every point.

By another non-trivial theorem (see [Mat]), all unipotent subgroups of $\mathrm{GL}_n(k)$ are conjugates of U_n. To give intuition for this statement, consider the case $n = 3$. Then the conjugates of U_3 under permutation matrices are:

$$\sigma = () : \quad \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix},$$

$$\sigma = (12) : \quad \begin{pmatrix} 1 & & * \\ * & 1 & * \\ & & 1 \end{pmatrix},$$

$$\sigma = (23) : \quad \begin{pmatrix} 1 & * & * \\ & 1 & \\ & * & 1 \end{pmatrix},$$

$$\sigma = (13) : \quad \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix},$$

$$\sigma = (123) : \quad \begin{pmatrix} 1 & & \\ * & 1 & * \\ * & & 1 \end{pmatrix},$$

$$\sigma = (132) : \quad \begin{pmatrix} 1 & * & \\ & 1 & \\ * & * & 1 \end{pmatrix}.$$

These are all the unipotent subgroups of $\mathrm{GL}_3(k)$. Notice that if we allow a subgroup to have any non-zero elements in both the $(1, 2)$ and $(2, 1)$ entries, then their conjugate will yield any element in the $(2, 2)$ entry. That is, a unipotent group cannot contain a subset of the form

$$\begin{pmatrix} 1 & * & \\ * & 1 & \\ & & 1 \end{pmatrix}.$$

Also, if a subgroup contains a subset of the form

$$\begin{pmatrix} 1 & & \\ * & 1 & \\ & * & 1 \end{pmatrix},$$

then it must contain all lower-triangular matrices.

2.2 Reductivity and Semisimplicity

Let $R(G)$ be the **radical** of a linear algebraic group G, that is defined as the maximal connected solvable normal subgroup of G. We let $R_u(G)$ be the **unipotent radical** of G, that is defined as the maximal connected unipotent normal subgroup of G. We call G **semisimple** if $R(G) = 0$, and **reductive** if $R_u(G) = 0$. As every unipotent group is solvable, we have $R_u(G) \subseteq R(G)$, so a semisimple group is reductive. If $\operatorname{char} k = 0$ then a group is reductive if and only if all finite-dimensional representations of it are semisimple.

The groups $\mathrm{GL}_n(k), \mathrm{SL}_n(k), \mathrm{O}(n,k), \mathrm{SO}(n,k), \mathrm{Sp}(n,k)$ are reductive. The multiplicative group $\mathbb{G}_m$ and finite products of it are reductive. The group $\mathrm{SL}_n(k)$ is furthermore semisimple, but $\mathrm{GL}_n(k)$ is not, as its radical is the subgroup of scalar matrices.

As non-examples, any unipotent group is not reductive, as its unipotent radical is itself. The Borel subgroup of $\mathrm{GL}_n(k)$, i.e. the subgroup of upper-triangular matrices, has U_n as its unipotent radical, so it is not reductive.

2.3 Simple Connectedness

A connected algebraic group G in characteristic zero is called **simply connected** if every isogeny $G' \to G$ is an isomorphism. Every semisimple algebraic group has a unique isogeny $\widetilde{G} \to G$ with $\widetilde{G}$ connected and simply connected.

The motivation for the definition is as follows. Suppose G is a simply connected Lie group, and $f : G' \to G$ is an isogeny (i.e. a Lie group homomorphism which is also covering map). Then the induced maps on the Lie algebras $\mathrm{d}f : \mathfrak{g}' \to \mathfrak{g}$ is an isomorphism, so it has an inverse $\mathrm{d}g$. As G is simply connected, $\mathrm{d}g$ can be integrated to a unique function $g : G \to G'$. By the same argument, $f \circ g = \mathrm{Id}_G$. Then, as $\mathrm{d}g : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism, G is a connected Lie subgroup of G', with g the inclusion map. But for any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}'$ there is a unique connected Lie subgroup of G' with Lie algebra $\mathfrak{h}$. Applying this to $\mathfrak{h} = \mathfrak{g} = \mathfrak{g}'$, and using the uniqueness part, we obtain $G \simeq G'$ via the map g. Then f is an isomorphism.

Conversely, if G is a Lie group such that each isogeny $G' \to G$ is an isomorphism, then G is simply connected as each Lie group has such an isogeny from its universal cover, which is simply connected.

2.4 Derived and Adjoint Subgroup

Assume G is an algebraic or a Lie group. The **derived** subgroup of G is the subgroup of G generated by all the commutators $ghg^{-1}h^{-1}$, and will be denoted by G^{der}. The **adjoint** group of G is the quotient of G by its centre, and will be denoted by G^{ad}. It is a non-trivial fact that the adjoint group of G is indeed an algebraic group. Note also that if k is not algebraically-closed, then the adjoint group is *not* given by the naïve quotient (it is the categorical quotient).

If G is reductive then G^{der} is semisimple.

2.5 Isogeny Classification of Semisimple Groups

We call a linear algebraic group **simple** if it has no non-trivial smooth, connected normal subgroups. This is known to be equivalent to having no infinite proper normal subgroups. Notice that this is weaker than the standard notion of simplicity of abstract groups. One intuition for this definition is that it defines a smooth algebraic group to be simple if and only if its Lie algebra is. Using this definition allows us to use Lie algebra theory, and the correspondence theorems between smooth algebraic groups and Lie algebras, to greater effect.

We have the following theorem, which we will not prove. A proof is given in [Milb][Theorem 4.5].

Theorem 2.1
An algebraic group is semisimple if and only if it is isogenuous to a product of simple algebraic groups.

3 Some Results

We now list some results above algebraic groups. Most of these can be found in [Mila].

3.1 More on Reductive Groups

For a reductive group G, we have the following commutative diagram:

$$\begin{array}{ccccc} & & G^{\mathrm{der}} & & \\ & & \downarrow & \searrow & \\ Z & \longrightarrow & G & \xrightarrow{\mathrm{ad}} & G^{\mathrm{ad}} \\ & \searrow & \downarrow{\scriptstyle \nu} & & \\ & & T & & \end{array}$$

Here, the column and row are short exact sequences, the diagnoal maps are isogenies with kernel $Z \cap G^{\mathrm{der}}$, the centre of G^{der}, and T is the abelianization of G, which is a torus. From this we obtain the short exact sequence

$$1 \to Z \cap G^{\mathrm{der}} \to Z \times G^{\mathrm{der}} \to G \to 1.$$

As an example, consider $G = \mathrm{GL}_n(k)$. Then the diagram is

$$\begin{array}{ccccc} & & \mathrm{SL}_n(k) & & \\ & & \downarrow & \searrow & \\ \mathbb{G}_m(k) & \longrightarrow & \mathrm{GL}_n(k) & \xrightarrow{\mathrm{ad}} & \mathrm{PGL}_n(k) \\ & {\scriptstyle (-)^n}\searrow & \downarrow{\scriptstyle \det} & & \\ & & \mathbb{G}_m(k) & & \end{array}$$

And the short exact sequence is

$$1 \to \mu_n \to \mathbb{G}_m(k) \times \mathrm{SL}_n(k) \to \mathrm{SL}_n(k) \to 1.$$

3.2 Tensors

Let G be a reductive group over a field k of characteristic zero. Let $\rho : G \to \mathrm{GL}_n(V)$ be a representation of G.

The **dual** $\rho^\wedge$ of ρ is the representation of G on the dual space $V^\wedge$ given by

$$(\rho^\wedge(g) \cdot f)(v) \coloneqq f(\rho(g^{-1}) \cdot v),$$

for all $g \in G, f \in V^\wedge, v \in V$. A representation is said to be **self-dual** if it is isomorphic to its dual.

An $\boldsymbol{r}$**-tensor** of V is an element of $(V^{\otimes r})^\wedge$. For an r-tensor t, the condition

$$t(gv_1, \dots, gv_r) = t(v_1, \dots, v_r) \qquad \forall (v_1, \dots, v_r) \in V^r,$$

cuts out an algebraic subgroup of $\mathrm{GL}(V)$, which we denote by $\mathrm{GL}(V)_t$. For a set of tensors T, the **subgroup of** $\mathbf{GL}(\boldsymbol{V})$ **fixing the** $\boldsymbol{t} \in \boldsymbol{T}$ is the subgroup $\bigcap_{t \in T} \mathrm{GL}(V)_t$.

Proposition 3.1
For any faithful self-dual representation $G \hookrightarrow \mathrm{GL}(V)$, there exists a finite set of tensor of V such that G is the subgroup of $\mathrm{GL}(V)$ fixing the $t \in T$.

Proposition 3.2
Let G be the subgroup of $\mathrm{GL}(V)$ fixing the tensors t in some set T. Then we have the isomorphism

$$\mathrm{Lie}(G) \simeq \{g \in \mathrm{End}(V) \mid \sum_j t(v_1, \dots, gv_j, \dots, v_r) = 0, \quad \forall t \in T, v_i \in V\}.$$

3.3 Real Points of an Algebraic Group

For an algebraic group G over $\mathbb{R}$, we denote by $G(\mathbb{R})^+$ the connected component of the identity element, taken in the real topology.

Proposition 3.3
If $\varphi : G \to H$ is a surjective map of algebraic groups over $\mathbb{R}$, then the induced map $\varphi : G(\mathbb{R})^+ \to H(\mathbb{R})^+$ is surjective.

Here the surjectivity is in the sense of maps of schemes. Note though that the map $G(\mathbb{R}) \to H(\mathbb{R})$ need not be surjective in general. As a counter-example, consider the surjective map $(-)^n : \mathbb{G}_m \to \mathbb{G}_m$; the induced map $(-)^n : \mathbb{R}^\times \to \mathbb{R}^\times$ is not surjective when n is even. Also, $\mathrm{SL}_2 \to \mathrm{PGL}_2$ is surjective, but the image of $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ is $\mathrm{PGL}_2(\mathbb{R})^+$.

Notice that if G is a simply connected algebraic group, then $G(\mathbb{C})$ is simply connected, but $G(\mathbb{R})$ need not be. As an example, $\mathrm{SL}_2(\mathbb{R})$ is not simply connected. This corresponds[2] to the universal cover of $G(\mathbb{R})$ not necessarily coming from an algebraic group. However, in some cases $G(\mathbb{R})$ is still nice enough.

Theorem 3.4
If G is a simply connected semisimple algebraic group over $\mathbb{R}$, then $G(\mathbb{R})$ is connected.

Corollary 3.5
If G is a reductive group over $\mathbb{R}$, then $G(\mathbb{R})$ has finitely many connected components.

[2] I think.

3.4 Cartan Involutions

Let G be a connected algebraic group over $\mathbb{R}$. An involution θ of G (i.e. an order two self-map of G as an algebraic group) is **Cartan** if the subgroup

$$G^{(\theta)}(\mathbb{R}) \coloneqq \{g \in G(\mathbb{C}) \mid g = \theta(\overline{g})\}$$

is compact, where $\overline{(-)}$ denotes complex conjugation.

Example 3.6
Suppose $G = \mathrm{SL}_2(\mathbb{R})$, and let $\theta = \mathrm{ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\theta\left(\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

So

$$\begin{aligned} \mathrm{SL}_2^{(\theta)}(\mathbb{R}) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \mid d = \overline{a}, c = -\overline{b}\right\} \\ &= \left\{\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid |a|^2 + |b|^2 = 1\right\} \\ &= \mathrm{SU}_2(\mathbb{R}). \end{aligned}$$

This is known to be compact, so θ is a Cartan involution.

Theorem 3.7
There exists a Cartan involution of G if and only if G is reductive. In that case, any two Cartan involutons of G are conjugate by an element of $G(\mathbb{R})$.

Example 3.8

(1) Notice that $G^{(\mathrm{Id}_G)}(\mathbb{C}) = G(\mathbb{R})$. Thus the identity map on G is a Cartan involution if and only if $G(\mathbb{R})$ is compact. By the theorem, in that case it is the only Cartan involution.

(2) Let $G = \mathrm{GL}(V)$ for V some real vector space. A choice of basis of V yields a transpose operator on G, and $M \mapsto (M^t)^{-1}$ is then a Cartan involution. As change of basis is given by conjugation, the theorem implies that all cartan involutions of G are given this way.

(3) Let $G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation of G. Then G is reductive if and only if it is stable under some transpose operator given by a choice of basis of V, and in that case the restriction of the map $M \mapsto (M^t)^{-1}$ in the previous example is a Cartan involution of G. It turns out that all Cartan involutions of G arise in this way.

(4) Let θ be an involution of G. There is a unique real form $G^{(\theta)}$ of $G(\mathbb{C})$ such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is given by $g \mapsto \theta(g)$. Thus θ is Cartan if and only if the real form $G^{(\theta)}$ is compact. All compact real forms of $G(\mathbb{C})$ arise in this way.

Proposition 3.9
If $G(\mathbb{R})$ is compact, then every finite-dimensional real representation of G has a G-invariant positive-definite symmetric bilinear form. Conversely, if one faithful real finite-dimensional representation of G has such a form, then $G(\mathbb{R})$ is compact.

The bilinear form is defined using Haar measure, which gives some intuition for why compactness matters.

Let $C \in G$ (strictly, $C \in G(\mathbb{R})$) whose square is central, which implies that $\mathrm{ad}(C)$ is an involution and $\mathrm{ad}(C) = \mathrm{ad}(C^{-1})$. A C-**polarization** on a real representation V of G is a G-invariant bilinear form φ such that the form

$$(u, v) \mapsto \varphi(u, Cv),$$

which we denote by φ_C, is symmetric and positive-definite.

Proposition 3.10
If $\mathrm{ad}(C)$ is a Cartan involution of G, then every finite-dimensional real representation of G has a C-polarization. Conversely, if some faithful finite-dimensional representation of G has a C-polarization, then $\mathrm{ad}(C)$ is a Cartan involution.

3.5 Approximation Theorems

Recall that we defined the **ring of finite adèles** as the restricted product

$$\mathbb{A}_f = \prod_{\ell} (\mathbb{Q}_\ell, \mathbb{Z}_\ell),$$

where ℓ runs over all the primes, and the restricted product means we require all elements (a_ℓ) to satisfy $a_\ell \in \mathbb{Z}_\ell$ for almost all ℓ. The topology is generated by products of opens, almost all of which are $\mathbb{Z}_\ell$.

Let G be an algebraic group over $\mathbb{Q}$. We call G of **compact type** if $G(\mathbb{R})$ is compact, and of **noncompact type** if it does not contain a non-trivial normal subgroup of compact type.

Theorem 3.11 (Strong Approximation)
If G is semisimple, simply connected, and of noncompact type, then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$.

Theorem 3.12 (Real Approximation)
If G is connected, then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$.

4 Weil Restriction

We now explain the concept of Weil restriction. Suppose L/K is a finite field extension, and X/L is a scheme over L. Consider the functor $(\mathbf{Sch/K})^{\mathrm{op}} \to \mathbf{Set}$ given by

$$T \mapsto X(T \times_K L).$$

If this functor is representable, then we call its representing scheme the **Weil restriction** of X to K, and we denote it by $\mathrm{Res}_{L/K} X$. With this definition one sees that Weil restriction takes group schemes to group schemes. Indeed, a group scheme may be defined as a scheme whose functor of points factors through $\mathbf{Grp}$, in which case almost by definition the functor of points of the Weil restriction factors through $\mathbf{Grp}$ as well.

We now limit ourselves to the case of X a variety over L. In this case, the Weil restriction of X to L exists, and is roughly given as follows. Suppose $X = L[y_1, \ldots, y_n]/(f_1, \ldots, f_m)$. Let $d = [L : K]$ and let $a_1, \ldots, a_d$ be a K-basis of L. Then we formally write

$$y_i = a_1 x_{i1} + \ldots + a_d x_{id}.$$

We now substitue this into the polynomials f_j, thus getting polynomials in the x-variables. However they still have L-coefficients. But now a relation of polynomial with L-coefficients can be written as d relations with K-coefficients, using the basis $a_1, \ldots, a_d$. We thus obtain the Weil restriction of X to K. We will not cover the explicit construction of the group structure here, in case X is a group variety.

We now demonstrate this via the so-called **Deligne torus**. It is the Weil restriction of $\mathbb{G}_m(\mathbb{C})$ to $\mathbb{R}$. We apply the above algorithm to it. Recall that $\mathbb{G}_m(\mathbb{C})$ was given by $\mathbb{C}[y_1, y_2]/(y_1 y_2 - 1)$. We write $y_1 = x_1 + ix_2, y_2 = x_3 + ix_4$. The relation $y_1 y_2 - 1 = 0$ then becomes

$$x_1x_3 - x_2x_4 - 1 = 0, \qquad x_1x_4 + x_2x_3 = 0.$$

So

$$\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \mathbb{R}[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2x_4 - 1, x_1x_4 + x_2x_3).$$

We claim that this is isomorphic to $\mathbb{R}[x, y][(x^2 + y^2)^{-1}]$, as one might expect the restriction of $\mathbb{G}_m(\mathbb{C})$ to be. The isomorphism is given by

$$x_1 \mapsto x, \quad x_2 \mapsto y, \quad x_3 \mapsto \frac{x}{x^2 + y^2}, \quad x_4 \mapsto -\frac{y}{x^2 + y^2}.$$

This is surjective as $(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1$. For injectivity, by multiplying an equation with the highest power of $x^2 + y^2$ in a denominator, it is enough to show injectivity on polynomials in x_1, x_2. But this is true as x_1, x_2 do not have any relations between themselves. Thus

$$\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \mathbb{R}[x, y][(x^2 + y^2)^{-1}].$$

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