Detection results for v_1-self maps at the prime 2

David Maurice Bowman

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Advisor: Dr. Jack Davies

Second Examiner: Prof. Dr. Markus Hausmann

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Abstract

The Periodicity Theorem in chromatic homotopy theory guarantees the existence of v_n-self maps on finite p-local spectra of type n. This thesis investigates the minimal periodicity of v_1-self maps at the prime 2. We utilise the BP-based Adams spectral sequence as well as develop the modern theory of modified Adams spectral sequences in order to compute with cofibres.

We apply these methods to detect v_1-self maps on quotients of the sphere spectrum $\mathbb{S}$ and the connective real K-theory spectrum ko. We recover the result that the mod 2 Moore spectrum $\mathbb{S}/2$ admits a v_1^4-self map but fails to admit a v_1^1-self map due to an obstruction given by η^2. We extend this analysis to $\mathbb{S}/4$, demonstrating the existence of a v_1^4-self map. Furthermore, we show that by forming the quotient $\mathbb{S}/(2, \eta)$, the obstruction vanishes, allowing for the construction of a v_1^1-self map. Finally, we analyse the spectrum $ko/2$ and its relation to ku and $k(1)$.

While the author does not know a reference for the results about $\mathbb{S}/(2, \eta)$ and $\mathbb{S}/4$ we certainly do not claim they are out of reach to anyone with access to the $\mathbb{E}_2$-page of the BP-based Adams spectral sequence for the sphere at $p = 2$ in the range $0 - 8 \times 0 - 8$.

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Chapter 0

Prerequisites, Conventions and Notation

We work in the setting of stable ∞-categories as developed by Lurie in [Lur09] and [Lur17]. Specifically, we let Sp denote the stable ∞-category of spectra and An denote the ∞-category of topological spaces. Unless otherwise stated, all limits and colimits are taken in the ∞-categorical sense. This framework allows us to manipulate limits and cofibres with the necessary homotopical flexibility, particularly in Section 2.3 where we develop the theory of modified Adams spectral sequences. All categories are assumed to be ∞-categories where the nerve functor is implicitly applied to all 1-categories.

We do not rely on the fine details of coherent multiplicative structures, hence we adopt the following conventions:

- By a **homotopy ring spectrum**, we mean a spectrum R that admits a monoid structure when considered as an object of hSp, the homotopy category of spectra.
- By a **homotopy commutative ring spectrum**, we mean a spectrum that admits a commutative monoid structure when considered as an object of hSp.

In the text we will often drop the work "homotopy" from these descriptions, however this will not cause any confusion as we will clearly indicate when using any facts about structured ring spectra.

0.1 Notation and Conventions

Throughout this thesis, we fix a prime p. In Chapter 3, we specialize to the case $p = 2$.

- **Localisation:** We denote the p-localisation of a spectrum X by $X_{(p)}$ and its p-completion by X_p.
- **Cofibres:** For a map $X \xrightarrow{f} Y$, we denote the cofibre by $C(f)$.
- **Adams Spectral Sequences:** We denote the functor $X \mapsto ASS_E(X)$, which associates a spectrum to its E-Adams spectral sequence, by ν.
- **Grading:** We index the Adams spectral sequence such that

$$E_2^{t-s,s} \cong \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \implies \pi_{t-s}(X).$$

Here s is the filtration degree and $t - s$ is the topological degree (the stem). In our charts, the horizontal axis represents the stem $(t - s)$ and the vertical axis represents the filtration (s).

Chapter 1

Introduction to Chromatic Homotopy Theory

In this section we give a brief overview of some salient aspects of chromatic homotopy theory as well as how this work fits into it. We largely follow the treatment of the subject given in the iteration of *Algebraic Topology II* taught by Dr. Jack Davies and Dr. Elizabeth Tatum in Bonn in the Summer of 2024. The reader familiar with the Ravenel conjectures may safely skip this section, however they are advised to look at Proposition **??** for motivation.

The primary goal of stable homotopy theory is to compute $\pi_*\mathbb{S}$, however, this turns out to be intractable with current methods. Chromatic homotopy theory proposes a divide and conquer approach that begins with the observation that for A an abelian group we have:

$$\begin{array}{ccc} A & \longrightarrow & \prod_p A_p \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Q} \otimes A & \longrightarrow & \mathbb{Q} \otimes \prod_p A_p \end{array} \quad .$$

We will define spectra $E(n)$ (depending on p) such that:

Theorem 1.1 (Chromatic Convergence Theorem)**.** *Let E be a finite p-local spectrum. Then E is the limit of*

$$\cdots \to L_{E(2)}E \to L_{E(1)}E \to L_{\mathbb{Q}}E.$$

We note p-complete implies p-local. This allows us to further reduce the study of $\mathbb{S}_p$ to $L_{E(n)}\mathbb{S}_p$ for all n. Later in this section we will discuss how to further divide this into smaller pieces via spectra $K(n)$.

1.1 Complex Oriented Spectra and Formal Group Laws

We begin by giving a brief overview of complex oriented spectra and formal group laws. Recall the space $BU(n)$ classifies n-dimensional complex line bundles, via pulling back γ_n the nth tautological bundle. We also note $BU(1)$ is modelled by $\mathbb{CP}^\infty$.

Definition 1.2. A **complex orientation** for a unital spectrum E is an element $x_E \in E^2(\mathbb{CP}^\infty)$ such that its restriction along $S^2 \simeq \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is the unit in $E^2(S^2) \simeq \pi_0 E$. A spectrum is **complex-orientable** if such an orientation exists.

This is equivalent to the unit $\mathbb{S} \to E$ factoring through the map

$$\mathbb{S} \simeq \Sigma^\infty \mathbb{CP}^1[-2] \to \Sigma^\infty \mathbb{CP}^\infty[-2].$$

Remark 1.3. Using the Atiyah-Hirzebruch spectral sequence one can show that for complex orientable E that

$$E^*(BU(n)) \simeq E_*[[x_1, \ldots, x_n]] \quad |x_i| = 2i.$$

Hence $E^*\mathbb{CP}^\infty \simeq E_*[[x_E]]$. This allows one to setup the theory of E-chern classes as done in the classical case for singular cohomology.

As an aide when looking for examples we have the following proposition from *Algebraic Topology II.*

Proposition 1.4. *Suppose $\pi_* E \simeq 0$ for $* \notin 2\mathbb{Z}$, then E is complex orientable.*

A complex oriented ring spectrum E comes with additional structure. There is a map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ which classifies the tensor product of line bundles. Following Remark 1.3, upon applying E^* we obtain

$$E^*[[x_E]] \to E^*[[x, y]].$$

Consider the image f_E of x_E under this map. It is a formal power series in 2 variables that is unital, commutative and associative (since m is). Furthermore, the algebra of formal power series allows us to construct for any $p(x)$ another powerseries with the unit as the constant term $q(x)$ such that $f_E(p(x), q(x)) = 0$. Hence f_E looks like a group structure over E^*. We codify this as:

Definition 1.5. A (one-dimensional, commutative) **formal group law** over a commutative ring R is a power series $f(x, y) \in R[[x, y]]$ such that:

a. $f(x, 0) = x$ and $f(0, y) = y$ (unitality).

b. $f(x, y) = f(y, x)$ (commutativity).

c. $f(x, f(y, z)) = f(f(x, y), z)$ (associativity).

Note we do not require inverses as part of the definition as the follow from the algebra of formal power series rings. Above we showed that complex oriented spectra admit formal group laws over their coefficient rings. The algebra of formal group laws shall turn out to be extremely useful.

Example 1.6. • For $H\mathbb{Z}$, $f_{H\mathbb{Z}}(x, y) = x + y$ (the additive formal group law). This follows because $m^*(x_\mathbb{Z}) = x_1 + x_2$ for the standard generator $x_\mathbb{Z} \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

• For KU, $f_{KU}(x, y) = x_1 + x_2 + u x_1 x_2$ (the multiplicative formal group law), where $u \in \pi_{-2} KU$ is the Bott element. This comes from

$$m^*(\gamma - 1) = (\gamma_1 - 1) + (\gamma_2 - 1) + (\gamma_1 - 1)(\gamma_2 - 1)$$

where γ is the universal line bundle, and $x_{KU} = (\gamma - 1)/u$.

We will need the notion of height of a formal group law to proceed. Let $f \in R[[x, y]]$ be a formal group law. Then we consider

$$[p]_f(x) \coloneqq f(x, f(x, ...)) \text{ iterated } p \text{ times.}$$

It follows from the definition that we can write

$$[p]_f(x) = px + \sum_{i\geq 1} v_i x^{p^i} + Q(x).$$

For later set the notation that $v_0 = p$.

Definition 1.7. With notation as above, we say f is of **height at least n at p** if $v_i = 0$ for $i < n$ and of **height exactly n at p** if additionally $0 \neq v_n$ is a unit in R.

There is a notion of morphism and isomorphism of formal groups laws and height is an invariant.
Suppose R has a formal group law f and $\phi : R \to S$ is a ring map, then $\phi_* f$, given by applying ϕ to the coefficients is a formal group law over S. Hence there is a covariant functor from rings to sets that takes a ring to the set of its formal group laws. This functor turns out to be corepresented by a ring:

Definition 1.8. The **Lazard ring** is $L := \mathbb{Z}[a_{i,j} | i, j \in \mathbb{N}]/I$ where I contains all the relations to enforce that $f_L := \sum_{i,j} a_{i,j} x^i y^j$ is a formal group law.

Then a map from $L \xrightarrow{\phi} R$ is the same as a formal group law over R via $\phi \mapsto \phi_* f_L$. It turns out L has a simpler model than one may expect:

Theorem 1.9 (Lazard)**.** *There is a noncanonical isomorphism of graded rings*

$$L \simeq \mathbb{Z}[x_i | i \in \mathbb{N}] \quad |x_i| = 2i.$$

There is an appropriately universal complex oriented spectrum which plays a key role in this subject:

Construction 1.10. *Consider the system $BU(1) \to BU(2) \to \dots$ given by classifying the tautological bundle plus a trivial summand. Let $T_n := Th(\gamma_n)$ be the Thom space of the nth tautological bundle. We note since $Th(\gamma \oplus \varepsilon) \simeq \Sigma^2 Th(\gamma)$ that $BU(n) \to BU(n+1)$ gives a map $T_n \to \Sigma^2 T_{n+1}$. Let $M(n) := \Omega^{2n} \Sigma^\infty T_n$. Then:*

$$MU := colim\,[M(0) \to M(1) \to M(2) \to \dots].$$

Note that $M(0) \simeq \mathbb{S}$ and $M(1) \simeq \Sigma^\infty \mathbb{CP}^\infty[-2]$, so by the universal property of colimits, MU is complex orientable. Now we shall observe some interesting properties enjoyed by MU:

Theorem 1.11 (Quillen)**.** $\pi_* MU \simeq L$.

This is known as *Algebraic Quillen's Theorem*. Next we note that MU can be given the structure of an $\mathbb{E}_\infty$-ring, described in [May77]. Hence a map of ring spectra $MU \to E$ gives a formal group law over E. We get even more: there is a functor $Or : CAlg(hSp) \to Set$ taking E to its set of complex orientations. Then *Topological Quillen's Theorem* says:

Theorem 1.12 (Quillen)**.** *There is a natural isomorphism of functors*

$$CAlg_{hSp}(MU, -) \simeq Or(-).$$

This establishes MU as a rather interesting spectrum with universal properties pertaining to both algebra and topology.
Now one can ask: *When does a ring map $L \xrightarrow{f} R$ lift to a map of spectra $MU \to E$ with $\pi_* E \simeq R$?* Somewhat surprisingly we get a good answer to this question by exploiting the algebra of formal group laws. We define a functor

$$\begin{aligned} E_*^f : An &\to grAb \\ X &\mapsto MU_*(X) \otimes_L R \end{aligned}$$

and now we want to know if this is a homology theory. Let's check the axioms: homotopy invariance follows from that of MU_* and the direct sum axiom follows as $\otimes_L$ commutes with colimits, hence we only need to verify exactness. This would be trivial if R was flat over L, but since L is an integral domain this would entail $L \hookrightarrow R$ which places significant constraints on R. By studying the algebraic geometry of the moduli stack of formal groups, Landweber was able to show:

Theorem 1.13 (Landweber Exact Functor Theorem)**.** *Let f be a formal group law over R. Then f lifts to a map of spectra $MU \to E^f$ if the sequence $(v_0, v_1, \dots)$ is a regular sequence in R for all p.*

Spectra arising in this way are called Landweber exact, and also enjoy some nice properties like the absence of phantom maps. As an example, KU can be constructed in this way: for the multiplicative formal group law f_m we have $[p]_{f_m}(x) = \frac{(1+ux)^p - 1}{u}$ which is congruent to $1/u$ mod p for all p. Hence for all p the sequence $(v_0, v_1, \dots)$ is $(p, 1/u, 0, 0, \dots)$. Since $\times p$ is injective on $\mathbb{Z}[u^{\pm}]$ and $1/u$ is a unit this sequence is regular.

1.2 Constructing Important Spectra

Fix a prime p. Now we shall use $MU_{(p)}$ to construct some spectra of specific interest. Since $MU_{(p)}$ is a structured ring spectrum we can form its category of modules. Then one has the following extension of scalars adjunction:

$$- \otimes MU_{(p)} : Sp \rightleftarrows Mod_{MU_{(p)}} : forget.$$

Hence a class $x_i : \Sigma^{2i}\mathbb{S} \to MU_{(p)}$ can also be represented as a map $\tilde{x_i} : \Sigma^{2i} MU_{(p)} \to MU_{(p)}$. By convention we declare $x_0 \coloneqq p$.

Proposition 1.14. *The cofibre of $\tilde{x_i}$ constructed above, $C(i)$ has homotopy groups given by $\pi_* MU_{(p)}/x_i$.*

Proof. We have that L is an integral domain, hence $\times x_i : L_{(p)} \hookrightarrow L_{(p)}$ is an injection of abelian groups. We consider the fibre sequence:

$$\Sigma^{2i} MU_{(p)} \xrightarrow{\times x_i} MU_{(p)} \to C(i)$$

and the resulting long exact sequence on homotopy groups:

$$\cdots \xrightarrow{\partial} \pi_{n-2i} MU_{(p)} \xrightarrow{\times x_i} \pi_n MU_{(p)} \to \pi_n C(i) \xrightarrow{\partial} \pi_{n-2i-1} MU_{(p)}.$$

By the injectivity of $\times x_i$ and the exactness of the sequence we deduce that all the boundary maps ∂ are zero, hence for each n we have a short exact sequence:

$$0 \to \pi_{n-2i} MU_{(p)} \xrightarrow{\times x_i} \pi_n MU_{(p)} \to \pi_n C(i) \to 0.$$

It follows that $\pi_n C(i) \simeq \pi_n(MU_{(p)})/\mathrm{im}(\times x_i)$ which yields the result.

□

By a similar argument to the above (again relying L being an integral domain) we get:

Proposition 1.15. *$\pi_*\left(\bigotimes_{i\in I} C(i)\right) \simeq \pi_* MU/(x_i | i \in I)$ for $I \subseteq \mathbb{N}$.*

Now we proceed:

Definition 1.16. We define the **Brown-Peterson spectrum** as $BP := \bigotimes_{i>0, i\neq p^n-1} C(i)$.

This spectrum admits an associative multiplicative structure. This is a smaller spectrum than $MU_{(p)}$ which contains the same information, justified by the following assertion at the beginning of Section 7.3 of [Rav92]:

Proposition 1.17. *$\langle MU_{(p)}\rangle = \langle BP\rangle$ where $\langle -\rangle$ denotes the Bousfield class of a spectrum.*

Hence when working p-locally it is sufficient to work with BP.

Definition 1.18. Let $I_n := \{p^j - 1 | j = 1, 2, ..., n\}$. Then we define the n**th Johnson-Wilson theory** as $E(n) := \bigotimes_{i>0, i\notin I_n} C(i)[v_n^{-1}]$.

This spectrum gives us the height less than or equal to n localisation, that is:

$$L_n := L_{E(n)}.$$

Definition 1.19. We define the n**th connective Morava K-theory**

$$k(n) := \left(\bigotimes_{k\neq p^n-1} C(k)\right)$$

and the n**th Morava K-theory** as $K(n) := k(n)[v_n^{-1}]$ where v_n is the class generating $\pi_{2(p^n-1)}k(n)$. We declare $K(0) := \mathbb{Q}$ and $K(\infty) := \mathbb{F}_p$.

These spectra admit associative multiplicative structures. Now we have an analogy to classical algebra. The fields $\mathbb{Q}, \mathbb{F}_p\ \forall p$ are the prime subfields over $\mathbb{Z}$ and can be thought of as the points of its spectrum. Similarly, one can show all fields in spectra (that is ring spectra such that all their modules are free) are modules over $K(n)$ for some n. Hence they are the analagous prime subfields over $\mathbb{S}$ and can be thought of as its points.

1.3 The Ravenel Conjectures

We can further our divide-and-conquer based approach to computing $\pi_*\mathbb{S}$ via:

Theorem 1.20. *For $X \in Sp$ there is a pullback square:*

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & \lrcorner & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}.$$

Using this in conjunction with the convergence theorem (Theorem 1.1) we reduce our problem to computing $\pi_* L_{K(n)}\mathbb{S}$ as well as the maps in the pullback diagram. Now we have a program where we study $K(n)$-local stable homotopy theory and use it to uncover insights into stable homotopy theory. We shall give some examples of this, stating some salient results from Chromatic Homotopy theory, conjectured in [Rav84] and later proved in [DHS88] and [HS98].

Theorem 1.21 (MU Nilpotence)**.** *Let R be a ring spectrum, then the kernel of the extension of scalars map $\pi_* R \to MU_* R$ consists of nilpotent elements.*

This allows us to give a slick proof of:

Theorem 1.22 (Nishida)**.** *Suppose $\alpha \in \pi_*\mathbb{S}$ such that $|\alpha| > 0$, then α is nilpotent.*

Proof. Recall $\pi_n\mathbb{S}$ is finite for $n \geq 0$ and hence each element is torsion in the ring. However such classes must vanish under $\pi_*\mathbb{S} \to MU_*\mathbb{S} \simeq L$ as L is torsion free. Now apply the Nilpotence theorem for MU. ☐

Furthermore, we see this is really a property of the family $\{K(n)\}_{n\in\mathbb{N}\cup\{0\}}$, we get:

Theorem 1.23 ($K(n)$ and BP Nilpotence)**.** *Let R be a p-local ring spectrum, and $\alpha \in \pi_* R$, then if either of the following hold:*

- *α maps to 0 under the extension of scalars map $\pi_* R \to BP_* R$,*

- *α maps to 0 under the extension of scalars map $\pi_* R \to K(n)_* R$ for all n,*

then α is nilpotent.

Definition 1.24. A spectrum $X \in Sp$ is called a **finite p-local spectrum** if X is in the essential image of the following composition:

$$Sp^{\omega} \hookrightarrow Sp \xrightarrow{-\otimes\mathbb{S}_{(p)}} Sp_{(p)}.$$

Some knowledge of this category can be obtained using these chromatic methods.

Proposition 1.25. *Let X be a finite p-local spectrum. Then $K(n)_* X \simeq 0 \implies K(m)_* X \simeq 0$ for $m \leq n$.*

Proof. We will use some commutative algebra and the theory of formal group laws to construct a spectrum which is acyclic for X and has $K(n-1)$ as a retract.

We consider $R = MU[v_n^{-1}] \otimes_{MU} \bigotimes_{k \neq p^n - 1, p^{n-1}-1} C(k)$ so that $\pi_* R \cong \mathbb{F}_p[v_{n-1}, v_n^{\pm}]$ and comes with surjection $\pi_* MU \to \pi_* R$. Hence R has a formal group law which we observe has height at least $n-1$.
Let (a, b) be the minimal positive solution to $a(2p^{n-1} - 2) - b(2p^{n-1} - 2) = 0$ then $\pi_0 R \cong \mathbb{F}_p[t]$ where $t = v_{n-1}^a v_n^{-b}$.
By construction there is a fibre sequence

$$\Sigma^{2p^{n-1}-2} R \xrightarrow{v_{n-1}} K(n).$$

After smashing with X and since $K(n)_* X \simeq 0$ by hypothesis, we obtain that v_{n-1} acts invertibly on $R_* X$ which implies t acts invertibly on $R_* X$. Next we observe that if F

is a free module over $\mathbb{F}_p[t]$ on which t acts invertibly then F is not finitely generated as one needs infinitely many generators for $t^{-1}, t^{-2}, \ldots$. Now, since X is finite, R_*X is a finitely generated $\mathbb{F}_p[t]$-module. Since we are working over a principal ideal domain, the classification of finitely generated modules tells us that R_*X must be torsion, else it would contain a free module and not be finitely generated.
Let $f(t) \in \mathbb{F}_p[t]$ denote a polynomial which kills R_*X (since R_*X is torsion consider its generators g_i and find elements f_i killing g_i respectively, then set $f(t) := \prod f_i$). We may choose $f(t)$ to be divisible by t. Now we can form $R[f(t)^{-1}]$ and note that smashing this with X yields 0.
Since $R[f(t)^{-1}]$ is again an MU-algebra it also carries a formal group law of height $n-1$ thus: $R[f(t)^{-1}] \otimes K(l) \simeq 0$ unless $l = n-1$ (in characteristic p formal group laws can only be isomorphic if they have the same height). Since we know $R[f(t)^{-1}]$ is nonzero we deduce $R[f(t)^{-1}] \otimes K(n-1)$ is nonzero.
Finally: $R[f(t)^{-1}] \otimes K(n-1)$ is a module over $K(n-1)$ and thus has $K(n-1)$ as a retract. We conclude by associativity of $\otimes$ that $K(n-1) \otimes X \simeq 0$. □

Now we let $\mathcal{C}_0$ denote the category of p-local finite spectra and for $n \geq 1$ let $\mathcal{C}_n \subseteq \mathcal{C}$ denote the full subcategory of $K(n-1)$-acyclic spectra. Then the above proposition gives us a diagram

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_\infty = 0. \tag{1.1}$$

We recall a full subcategory $\mathcal{C} \subseteq \mathcal{C}_0$ is *thick* if it is closed under (co)fibres, direct sums and retracts.

Theorem 1.26 (Thick Subcategory Theorem)**.** *Suppose $\mathcal{C} \subseteq \mathcal{C}_0$ is a thick subcategory then $\mathcal{C} = \mathcal{C}_n$ for some $n \in \mathbb{N} \cup \{0, \infty\}$.*

We note that in this setting thick subcategories are tensor ideals, here we see that kernels of $L_{K(n)}$ are all thick subcategories.

Given the thick subcategory theorem, one may ask if these inclusions are strict. It turns out they are! We shall now introduce our main objects of study for this thesis:

Definition 1.27. A spectrum $X \in \mathcal{C}_0$ is of **type at least** n if $K(n-1)_*X \simeq 0$ and of **type** n if $K(n)_*X \not\simeq 0$ and $K(n-1)_*X \simeq 0$.

Definition 1.28. A self map $f : \Sigma^l X \to X$ is called a v_n**-self map** if it acts as an automorphism on $K(n)_*X$ and acts as 0 on $K(m)_*X$ for all $m \neq n$.

Since $K(n)$ can always be given a homotopy associative multiplicative structure, $K(n)_*X$ has the structure of a graded module over $\mathbb{F}_p[v_n^{\pm}]$. Then it is shown in Lemma 6.1.1. of [Rav92] that:

Proposition 1.29. *For a v_n-self map $f : \Sigma^l X \to X$ there exists $i \geq 0$ such that f^i acts as v_n^j on $K(n)_*X$ for some $j > 0$.*

Hence in our quest to detect v_n-self maps, it will be sufficient to understand maps which eventually act by v_n to some power. The alert reader would perhaps want a spectrum to be of type n before one considers v_n-self maps, however we have:

Proposition 1.30. *Suppose $X \not\simeq 0$ admits a v_n-self map f, then X is of type n.*

Proof. Consider

$$\Sigma^l X \xrightarrow{f} X \to C(f).$$

First we apply $K(n)_*$ and observe $K(n)_*C(f) \simeq 0$ since f induces an isomorphism on $K(n)_*$, hence $K(m)_*C(f) \simeq 0$ and this implies $K(m)_*f$ is an isomorphism, but it is also 0, hence X is of type n. □

We get even more, for $m > n$ since $K(m)_* f = 0$ one has

$$K(m)_* X \simeq K(m)_* C(f) \not\simeq 0$$

so given a v_n-self map we can produce a type $n+1$ spectrum!

Theorem 1.31 (Periodicity Theorem)**.** *Suppose $X \in \mathcal{C}_0$ is of type n, then X admits a v_n-self map.*

This tells us there are spectra of every type and that the inclusions in (1.1) are strict. The proof of this theorem involved the development of tools to detect v_n-self maps using the Adams spectral sequence, in particular the proof also shows that the power of v_n that acts must be a power of p. In this thesis we shall use these tools to deduce the existence and nonexistence of some v_n-self maps. Outside of the Periodicity Theorem there is another reason to be interested in v_n-self maps.

The *telescope conjecture*, reformulated in [MR01] asserts that for any type n p-local finite spectrum X, there is an equivalence[1]:

$$X[v_n^{-1}] \simeq L_n X.$$

A counter example was recently constructed in [Bur+23], showing:

Theorem 1.32. *The telescope conjecture is false for $n \geq 2$.*

The most direct approach to addressing this conjecture would be to write down type n p-local finite spectra X, compute both $X[v_n^{-1}]$ and $L_n X$ and try to show they are different spectra. To carry this out, one must have access to examples of v_n-self maps.

In Chapter 2 we discuss the methods we will use to produce examples of v_n-self maps, culminating in Section 2.3 where we show how to produce a *modified Adams spectral sequence* for X/e where X is a finite spectrum and $e : \Sigma^{|e|} X \xrightarrow{\times e} X$ is a homotopy element that lifts to a self map. Then in Chapter 3 we specialise to the case of $p = 2$ and $n = 1$ and use our techniques to produce examples of v_1-self maps for quotients of the sphere, ko and ku.

[1] By $X[v_n^{-1}]$ we mean the result of inverting the smallest power of v_n acting on $K(n) \otimes X$ arising from a self map of X.

Chapter 2

Techniques and Tools

We shall begin with a brief overview of spectral sequences, then discuss the Adams spectral sequence as well as the formalism we shall use to work with it - that of filtered spectra. For our discussion of spectral sequences we shall follow the iteration of *Algebraic Topology I* taught by Professor Markus Hausmann in Bonn in the Winter of 2023.

Definition 2.1. A **spectral sequence** consists of tuples $(\mathbb{E}_r, d_r, h_r)_{r\geq 2}$ such that:

- $\mathbb{E}_r$ is a bigraded abelian group,
- $d_r : \mathbb{E}_r^{*,*} \to \mathbb{E}_r^{*-1,*+r}$ is a differetial (so $d_r \circ d_r = 0$) of bidegree $(-1, r)$,
- h_r is a bigrading preserving isomorphism

$$H^*(\mathbb{E}_r, d_r) \simeq \mathbb{E}_{r+1}.$$

If there exists an R such that for all $r \geq R$ it holds that $\mathbb{E}_r \simeq \mathbb{E}_R$ then the spectral sequence has an $\mathbb{E}_\infty$-page and $\mathbb{E}_\infty := \mathbb{E}_r$ for and $r \geq R$.

A **multiplicative structure** on a spectral sequence $(\mathbb{E}_r, d_r, h_r)_{r\geq 2}$ consists of maps

$$\mathbb{E}_r^{p,q} \otimes \mathbb{E}_r^{p',q'} \to \mathbb{E}_r^{p+p',q+q'}$$

such that for $x \in \mathbb{E}_r^{p,q}$ we have:

$$d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y) \tag{2.1}$$

as well as a refinement of each h_r to an isomorphism of bigraded rings. The relation 2.1 is called the **Leibniz rule**.

A multiplicative spectral sequence $(\mathbb{E}_r, d_r, h_r)_{r\geq 2}$ is said to **converge** to a filtered graded ring $(R_\bullet, F)$ if the $\mathbb{E}_\infty$-page exists as well as isomorphisms

$$\mathbb{E}_\infty^{p,q} \simeq F^q R_p / F^{q+1} R_p$$

compatible with the multiplicative structures.

We shall carry out some example computations. First we recall:

Theorem 2.2 ([Ser51])**.** *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibre sequence of spaces, then there exists a multiplicative spectral sequence

$$\mathbb{E}_2^{p,q} := H^p(B; H^q(F; \mathbb{Z})) \implies H^{p+q}(E; \mathbb{Z})$$

with $|d_r| = (r, 1-r)$.

Let X be a space and consider a map $* \to X$, then by checking on the relevant long exact sequence on homotopy groups the homotopy fibre is deduced to be ΩX. Hence for a space X we get a Serre spectral sequence associated to:

$$\Omega X \to * \to X.$$

It holds that $H^n(*) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$. Hence in such a spectral sequence we would expect differentials to kill everything on the $\mathbb{E}_2$-page away from $(0, 0)$.

Proposition 2.3. *It holds that* $H^n(\Omega S^k) \simeq \begin{cases} \mathbb{Z} & \textit{if } (k-1) \mid n \\ 0 & \textit{else} \end{cases}$ *for* $k \geq 2$.

Proof. We shall do the proof for $k = 3$ and it will be clear how to generalise to other k. We consider the fibre sequence

$$\Omega S^3 \to * \to S^3$$

and its Serre spectral sequence:

$$H^p(S^3, H^q(\Omega S^3)) \implies H^{p+q}(*).$$

We recall $H^n(S^3) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0, 3 \\ 0 & \text{else} \end{cases}$ and thus we only have nonzero classes in the $0th$ and $3rd$ columns, each with a $\mathbb{Z}$ at the base. Due to convergence we know the class in degree $(3, 0)$ does not survive to the $\mathbb{E}_\infty$-page, hence it must support a differential. The only possible nonzero differential is d_3, hence $\mathbb{E}_2^{0,2} \simeq \mathbb{Z}$ and $\mathbb{E}_3^{0,2} \xrightarrow{d_3} \mathbb{E}_3^{3,0}$ is an isomorphism. Hence $\mathbb{E}_2^{3,2} \simeq H^2(\Omega S^3) \simeq \mathbb{Z}$ and the argument repeats, necessitating a class in degree $(0, 4)$ to kill it. Repeating this argument indefinitely yields the result. The Serre spectral sequence is shown in Figure 2.1, where each $\bullet$ is a copy of $\mathbb{Z}$.

□

Using the same fibre sequence and similar arguments one can also show the following:

Proposition 2.4. *Let X be a simply connected space, then one of the following is false:*

- *X is modelled by a finite CW-complex,*
- *ΩX is modelled by a finite CW-complex.*

2.1 Filtered Spectra

As we will be working with quotients of spectra with known Adams $\mathbb{E}_2$-pages it will be useful to have a "category of spectral sequences" which is amenable to algebraic manipulations. We will follow [Nig25].

Definition 2.5. A **filtered spectrum** is a functor $X : \mathbb{Z}^{op} \to Sp$. The category of filtered spectra is

$$FilSp := Fun(\mathbb{Z}^{op}, Sp).$$

We recall that Proposition 1.1.3.1 of [Lur17] asserts that functor categories with stable codomains are stable, hence we conclude that $FilSp$ is also stable. This gives us the algebraic structure we are looking for.

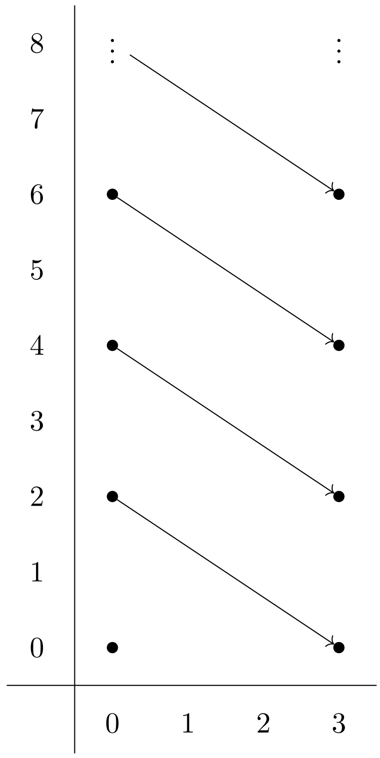


Figure 2.1: Serre spectral sequence for $\Omega S^3 \to * \to S^3$

Furthermore we can endow $FilSp$ with a symmetric monoidal structure given by Day convolution, it is given by:

$$(X \otimes Y)_k := \operatorname*{colim}_{i+j \leq k} (X_i \otimes_{\mathbb{S}} Y_j)$$

where we consider $\mathbb{Z}^{op}$ with the symmetric monoidal structure given by addition and Sp with the symmetric monoidal structure given by the smash product. We note that stability means we can make sense of cofibres which will be explored in section 2.3. It also grants us a suspension functor given by the following pushout:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array} .$$

We note that these suspensions are computed pointwise as colimits in functor categories are computed pointwise.

We note that $FilSp$ has a t-structure where the connective objects are given by X such that X^n is n-connective, it is called the *diagonal t-structure*. Furthermore, the connective cover is given by:

$$(\tau_{\geq 0}^{diag} X)^n := \tau_{\geq n} X^n.$$

As we are working in the setting of infinity categories where functors are difficult to write down, one must be careful when producing examples of filtered spectra.

We recall that for any category $\mathcal{C}$ it holds that $Fun(\Delta^0, \mathcal{C}) \simeq \mathcal{C}$ given by evaluation. Hence, we may consider $Fun(\Delta^0, Sp)$ as a model for Sp.

Example 2.6. For $X \in Sp$:

- There is a unique functor $\mathbb{Z}^{op} \to \Delta^0$ and by precomposition this gives rise to a functor Const : $Sp \to FilSp$ which is given on objects by:

$$X \mapsto (\cdots \to X \to X \to X \to \cdots),$$

- Using the diagonal t-structure above we can produce the Whitehead tower Wh $:= \tau_{\geq 0}^{diag} \circ$ Const : $Sp \to FilSp$ which is given on objects by:

$$X \mapsto (\cdots \to \tau_{\geq 1}X \to \tau_{\geq 0}X \to \tau_{\geq -1}X \to \cdots),$$

- Similarly to Wh, we can define the Postnikov tower Post $:= \tau_{\leq 0}^{diag} \circ$ Const : $Sp \to FilSp$ which is given on objects by:

$$X \mapsto (\cdots \to \tau_{\leq 1}X \to \tau_{\leq 0}X \to \tau_{\leq -1}X \to \cdots).$$

Now we shall construct a functor $\mathbb{Z}^{op} \to FilSp$ which will allow us to give a generating set for $FilSp$. We recall that $\mathrm{Hom}_{\mathbb{Z}^{op}}(s,t) = \begin{cases} * & \text{if } s \geq t \\ \emptyset & \text{else} \end{cases}$. We can consider the output of this functor as discrete Anima and hence it makes sense to compose it with $\Sigma^\infty_+ : An \to Sp$. Hence we get a functor

$$\begin{aligned} i : \mathbb{Z}^{op} &\to FilSp \\ s &\mapsto \Sigma^\infty_+ \circ \mathrm{Hom}_{\mathbb{Z}^{op}}(-,s). \end{aligned}$$

Definition 2.7. a. The **filtered (s,f)-sphere** is defined as:

$$\mathbb{S}^{s,f} := \Sigma^s i(f).$$

We refer to s as the **stem** and f as the **filtration**.

b. We define $\Sigma^{s,f} := \mathbb{S}^{s,f} \otimes - : FilSp \to FilSp$,

c. We define $\pi_{s,f} : FilSp \to Ab$ as $[\mathbb{S}^{s,f}, -]$.

Unpacking these definitions we observe that:

$$\mathbb{S}^{s,f} = (\cdots \to 0 \to \mathbb{S}^s \to \mathbb{S}^s \to \cdots)$$

with the first $\mathbb{S}^s$ appearing in position f. Furthermore, applying $\Sigma^{s,f}$ is the same as applying Σ^s levelwise and shifting the diagram f places left. Finally, one also observes a natural isomorphism $\pi_{s,f}X \simeq \pi_s X^f$.

Proposition 2.8. *The category $FilSp$ is generated by $S^{0,f}$ under colimits.*

Proof. This is Proposition 2.21 in [Nig25]. □

We shall now show how to produce a spectral sequence given $X \in FilSp$:

Definition 2.9. We define the category of **graded Spectra** as

$$grSp := \prod_{\mathbb{Z}} Sp \simeq Fun(\mathbb{Z}^\delta, Sp)$$

where $\mathbb{Z}^\delta$ is the discrete category with objects $\mathbb{Z}$. Furthermore, we define the category of **bigraded Spectra** as

$$gr^2Sp := Fun(\mathbb{Z}^\delta, grSp) \simeq Fun(\mathbb{Z}^\delta \times \mathbb{Z}^\delta, Sp).$$

Using the description as a functor category, we once again apply Proposition 1.1.3.1 of [Lur17] to obtain that $grSp$ is a stable category. We also have functors

$$\begin{aligned} \pi_{s,f} : grSp &\to Ab \\ X &\mapsto \pi_s X^f \end{aligned}$$

which assemble into a functor $\pi_{*,*} : grSp \to gr^2 Ab$ with one grading for the homotopy groups and one for the filtration grading.

We can define a functor $Gr : FilSp \to grSp$ given on objects by

$$Gr^s X := \mathrm{cofib}(X^{s+1} \to X^s).$$

This assembles into a functor. A map of filtered spectra $X \xrightarrow{f} Y$ is the data of maps $X^s \xrightarrow{f^s} Y^s$ as well homotopies witnessing the commutativity of the f^s with the structure maps. This allows us to construct maps $Gr^s X \to Gr^S Y$ and the functoriality follows from the universal property of colimits.
Now we may construct our spectral sequences: We denote by $X^{-\infty}$ the colimit of X and consider X as a tower:

$$\cdots \to X^{n+1} \xrightarrow{\phi^n} X^n \xrightarrow{\phi^{n-1}} X^{n-1} \to \cdots .$$

For each n, we can form cofibre sequence $X^{n+1} \xrightarrow{\phi^n} X^n \xrightarrow{\psi^n} X^n_n$ which yields a boundary map $X^n_n \xrightarrow{\partial^n} \Sigma X^{n+1}$. Now we can apply $\pi_{*,*} \circ Gr$ and produce:

$$\begin{array}{ccc} \bigoplus_{s,t} \pi_{t-s} X^{t+1} & \xrightarrow{\quad f \quad} & \bigoplus_{s,t} \pi_{t-s} X^t \\ & {\scriptstyle \partial} \nwarrow \quad \swarrow {\scriptstyle g} & \\ & \bigoplus_{s,t} \pi_{t-s} X^t_t & \end{array}$$

where both ϕ, ψ have bidgree $(0,0)$ and ∂ has degree $(0,-1)$. One can check this is an exact couple, and by following section 2.2 of [McC01] we obtain a spectral sequence:

$$E_2^{t-s,s}(X) := \pi_{t-s} X^t_t \implies \pi_{t-s} X^{-\infty}$$

with differentials d_r of bidegree $(-1, r)$.

2.2 The Adams Spectral Sequence

The Adams spectral sequence is an extremely useful tool in stable homotopy theory, particularly for computing $\pi_* \mathbb{S}$. It was first introduced by Adams in [Ada58] where he uses it to solve the Hopf Invariant One problem. Today there is a vast body of theory regarding the Adams spectral sequence. For our purposes a short, practical introduction to this spectral sequence will do. We will specialise to the cases where one takes either $\mathbb{F}_p$, BP, or MU as a base and only consider it for connective spectra which will greatly simplify our discussion of convergence. In this section we follow [Rav86] and [Nig25].

Let $E \in \{\mathbb{F}_p, BP\}$ and X a connective spectrum, then the E-based Adams Spectral Sequence is a multiplicative spectral sequence with signature:

$$\mathbb{E}_2^{t-s,s} := \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \implies \pi_{t-s}X_{\widehat{E}} \tag{2.2}$$

where d_r has degree $(-1, r)$ and by Theorem 2.2.13 of [Rav86] we have:

$$X_{\widehat{E}} = \begin{cases} X_{(p)} & \text{if } E = BP \\ X_p & \text{if } E = \mathbb{F}_p \end{cases}.$$

We refer to $X_{\widehat{E}}$ as the *E-nilpotent completion of X* and if $X \simeq X_{\widehat{E}}$ then X is *E-nilpotent complete*.

This grading convention leads to the sth stem with its filtration being given in the sth column which is convenient for computations.

Before proceeding with the definitions and constructions, we would like to draw attention to the excellent work done in [LWX24] - they have published a website[1] where one may access the $\mathbb{F}_2$-based Adams $\mathbb{E}_2$-pages for many commonly considered spectra.

First, we shall make clear what we mean by this Ext group that appears on the $\mathbb{E}_2$-page. We shall discuss a natural structure that is enjoyed by (E_*, E_*E) that will allow us to define a certain abelian category. Then, we will observe that E^* gives a functor into this abelian category where Ext can be computed.

Definition 2.10 ([Rav86] A1.1.1.)**.** A **Hopf algebroid** over a commutative ring K is a cogroupoid object in the category of (graded or bigraded) commutative K-algebras, i.e., a pair (A, Γ) of commutative K-algebras with structure maps such that for any other commutative K-algebra B, the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid (a small category in which every morphism is an equivalence). The structure maps are

$$\begin{aligned} \eta_L : A &\to \Gamma && \text{left unit or source,} \\ \eta_R : A &\to \Gamma && \text{right unit or target,} \\ \Delta : \Gamma &\to \Gamma \otimes_A \Gamma && \text{coproduct or composition,} \\ \varepsilon : \Gamma &\to A, && \text{counit or identity,} \\ c : \Gamma &\to \Gamma && \text{conjugation or inverse.} \end{aligned}$$

Here Γ is a left A-module via η_L and a right A-module via η_R, $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules, and Δ and ε are A-bimodule maps. The defining properties of a groupoid correspond to the following relations among the structure maps:

(a) $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$, the identity map on A. (The source and target of an identity morphism are the object on which it is defined.)

(b) $(\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes \Gamma)\Delta = 1_\Gamma$. (Composition with the identity leaves a morphism unchanged.)

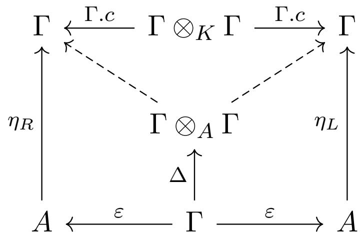
(c) $(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta$. (Composition of morphisms is associative.)

(d) $c\eta_R = \eta_L$ and $c\eta_L = \eta_R$. (Inverting a morphism interchanges source and target.)

(e) $cc = 1_\Gamma$. (The inverse of the inverse is the original morphism.)

[1] https://waynelin92.github.io/ss/kervaire-49.html

(f) Maps exist which make the following commute



where $c\cdot\Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$. (Composition of a morphism with its inverse on either side gives an identity morphism.)

It is asserted by Proposition 2.2.8. of [Rav86] that (E_*, E_*E) admits a Hopf algebroid structure. We shall spell out the structure maps: η_L is given by tensoring the unit $\mathbb{S} \to E$ with E on the left, similarly η_R is given by tensoring with E on the right. We denote by $\mu : E \otimes E \to E$ the multiplication on E, this induces the counit $\varepsilon : E_*E \to E_*$. We consider:

$$(E \otimes E) \otimes (E \otimes X) \xrightarrow{1\otimes\mu\otimes 1} E \otimes E \otimes X.$$

Lemma 2.11 ([Rav86] 2.2.7.)**.** *The above map induces an isomorphism*

$$E_*E \otimes_{E_*} E_*X \simeq \pi_*(E \otimes E \otimes X).$$

Now the map

$$E \otimes X \simeq E \otimes \mathbb{S} \otimes X \to E \otimes E \otimes X$$

induces

$$\psi : E_*(X) \to E_*E \otimes_{E_*} E_*X,$$

by choosing $X = E$ we obtain a map

$$\Delta : E_*E \to E_*E \otimes_{E_*} E_*E.$$

Finally, the map $c : E_*E \to E_*E$ is induced by switching the factors of $E \otimes E$.

The reason we defined Hopf algebroids is so we can define:

Definition 2.12 ([Rav86] A1.1.2)**.** A **left Γ-comodule** M is a left A-module M together with a left A-linear map $\psi : M \to \Gamma \otimes_A M$ which is counitary and coassociative, i.e., such that $(\varepsilon \otimes M)\psi = M$ (i.e., the identity on M) and $(\Delta \otimes M)\psi = (\Gamma \otimes \psi)\psi$. A right Γ-comodule is similarly defined. An element $m \in M$ is primitive if $\psi(m) = 1 \otimes m$.

A comodule algebra M is a comodule which is also a commutative associative A-algebra such that the structure map ψ is an algebra map. If M and N are left Γ-comodules, their comodule tensor product is $M \otimes_A N$ with structure map being the composite

$$M \otimes N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes M \otimes \Gamma \otimes N \to \Gamma \otimes \Gamma \otimes M \otimes N \to \Gamma \otimes M \otimes N,$$

where the second map interchanges the second and third factors and the third map is the multiplication on Γ. All tensor products are over A using only the left A-module structure on A.

Then we have:

Theorem 2.13 ([Rav86] A1.1.3. and A1.2.2)**.** *For E as above, the category of left E_*E-comodules is abelian and has enough injectives.*

Our construction of the map $\psi : E_*(X) \to E_*E \otimes_{E_*} E_*X$ above implies that

$$E_* : Sp \to grAb$$

actually factors through E_*E- comodules. Hence the Ext groups in 2.2 are just Ext in the category E_*E-comodules between $E_*\mathbb{S} \simeq E_*$ and $E_*(X)$ which have comodule structures described above.

Definition 2.14. A **cosimplicial spectrum** is a functor $\Delta \to Sp$ where Δ is the category of finite ordered sets and nondecreasing maps.

Now that we have all the definitions in place, we can go over a brief construction of the E-based Adams spectral sequence as carried out in [Nig25]. We shall build a cosimplicial spectrum and then consider the filtered spectrum given by its filtration by partial totalisations, this will give rise to the Adams spectral sequence.

Definition 2.15 ([Nig25] 2.68)**.** Let $X^\bullet$ be a cosimplicial spectrum, then the **totalisation** of X is given by

$$\mathrm{Tot}(X^\bullet) := \lim_{\Delta} X^\bullet.$$

The n**th partial totalisation** is given by

$$\mathrm{Tot}_n(X^\bullet) := \lim_{\Delta_{\leq n}} X^\bullet.$$

By the universal property of limits we get a map

$$\mathrm{Tot}(X^\bullet) \xrightarrow{p_n} \mathrm{Tot}_{n-1}(X^\bullet)$$

and we define $\mathrm{Tot}^n(X^\bullet) := \mathrm{fib}(p_n)$. Then we can define a **totalisation filtration** of $X^\bullet$:

$$\mathrm{filTot}(X^\bullet) := \left(\cdots \to \mathrm{Tot}^2(X^\bullet) \to \mathrm{Tot}^1(X^\bullet) \to \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(X^\bullet) \to \cdots \right).$$

Definition 2.16. The unit map $\mathbb{S} \to E$ will give rise to an augmented cosimplicial spectrum:

$$\mathbb{S} \longrightarrow E \rightrightarrows E\otimes \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

The **cosimplicial** E**-based Adams resolution** of X is

$$ASS_E^{\Delta}(X) := E^{[\bullet]} \otimes X = \left(E \otimes X \rightrightarrows E \otimes E \otimes X \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

Now we define the filtered spectrum

$$ASS_E(X) := \mathrm{filTot}(ASS_E^{\Delta}(X))$$

and we shall see in the next section how this gives rise to the E-based Adams spectral sequence. We note here that

$$\mathrm{colim} ASS_E(X) \simeq X_{\widehat{E}}$$

as this is essentially the definition of $(-)_{\widehat{E}}$ and hence if X is E-nilpotent complete then

$$\mathrm{colim} ASS_E(X) \simeq X.$$

Remark 2.17. For posterity, we cite [Nig25] Remark 4.74 to mention that $ASS_E(-)$ is functorial and if the base E admits an $\mathbb{E}_2$ structure or better (as is the case for $E \in \{\mathbb{F}_p, BP\}$) then $ASS_E(X)$ will admit an $\mathbb{E}_1$ structure and hence give us a multiplicative spectral sequence.

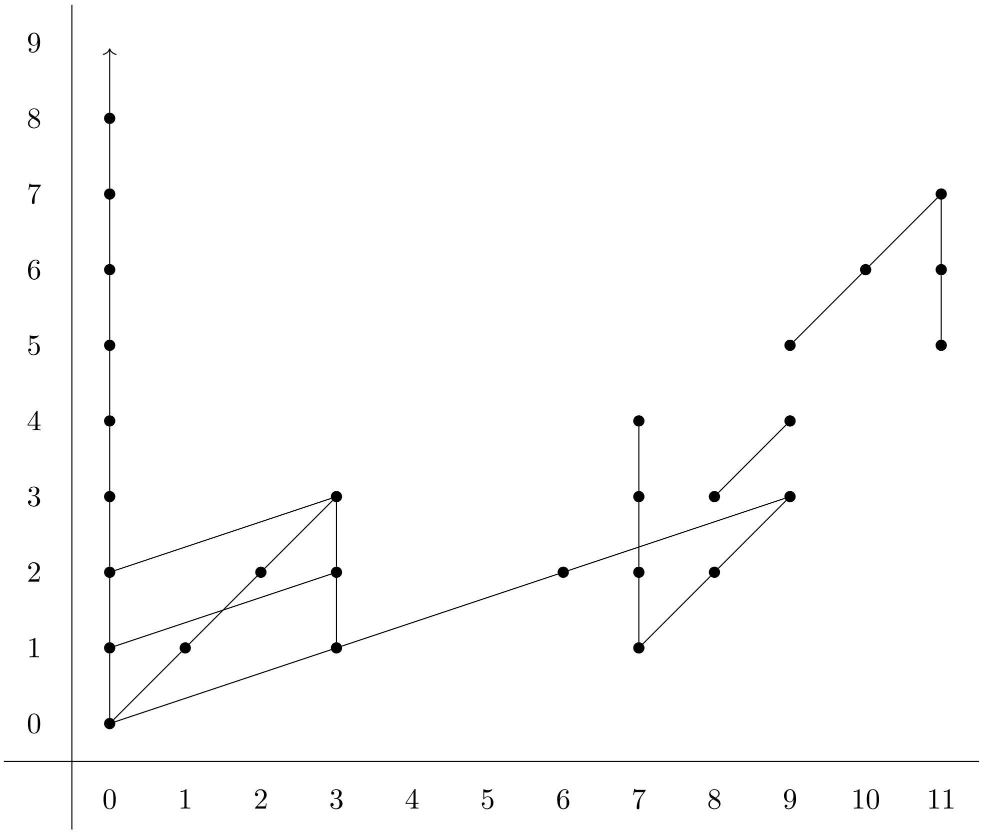


Figure 2.2: $\mathbb{F}_2$-based Adams spectral sequence for $\mathbb{S}_2$

Notation 2.18. For notational convenience we shall use the symbol $\nu : Sp \to FilSp$ to denote the functor ASS_E, where E will be clear from context.

Now we shall look at the $\mathbb{E}_2$-pages of the $\mathbb{F}_2$ and $\mathbb{F}_3$-based Adams spectral sequence for $\mathbb{S}$ and see what we can learn.

In Figure 2.2 each $\bullet$ is a copy of $\mathbb{F}_2$. In low degrees there are no differentials and we can read some of $\pi_*\mathbb{S}_2$ off the chart, they are in Table 2.1. The vertical multiplication lines are given by $\times h_0$ where h_0 is the permanent cycle that detects $2 \in \pi_*\mathbb{S}_2$. These lines allow us to see 2-extensions and for example allow us to see that $\pi_3\mathbb{S}_2 \simeq \mathbb{Z}/8\mathbb{Z}$ and not $\mathbb{F}_2^3$. We will come to discuss why $|h_0| = (0,1)$, that is why it has filtration 1. We should draw attention to some special classes on the $\mathbb{E}_2$-page: h_1 is the generator in degree $(1,1)$ which detects the generator $\eta \in \pi_1\mathbb{S}_2$. There is also h_2 of degree $(3,1)$ which detects the generator $\nu \in \pi_3\mathbb{S}_2$. We read the relation $h_1^3 = h_0^2 h_2$ straight off the chart. We can also infer that any class without a class one filtration above it is necessarily h_0-torsion, and hence if it is a permanent cycle it detects a 2-torsion element. We can make similar inferences about torsion for other classes, for example a class without a class one stem and filtration higher is necessarily h_1 torsion and so on. There is also h_3 of degree $(7,1)$ which detects the generator $\sigma \in \pi_7\mathbb{S}_2$. In general there are classes h_i of degree $(1, 2^i - 1)$, but Adams showed [Rav78] for $i \geq 4$ that $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ and hence h_i can only be a permanent cycle if $i \leq 3$.

Now we carry out a similar analysis of the Figure 2.3 to study $\mathbb{S}_3$. The reader will notice the $\mathbb{F}_3$-based $\mathbb{E}_2$ page is much simpler, at least in small degrees. In this case each $\bullet$ is a copy of $\mathbb{F}_3$. Once again, in low degree we can read $\pi_*\mathbb{S}_3$ right off the chart, this is displayed in Table 2.2.

Index	0	1	2	3	4	5	6	7	8	9
Group	$\mathbb{Z}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{Z}/8\mathbb{Z}$	0	0	$\mathbb{F}_2$	$\mathbb{Z}/16\mathbb{Z}$	$\mathbb{F}_2^{\oplus 2}$	$\mathbb{F}_2^{\oplus 3}$

Table 2.1: $\pi_*\mathbb{S}_2$ in low degrees

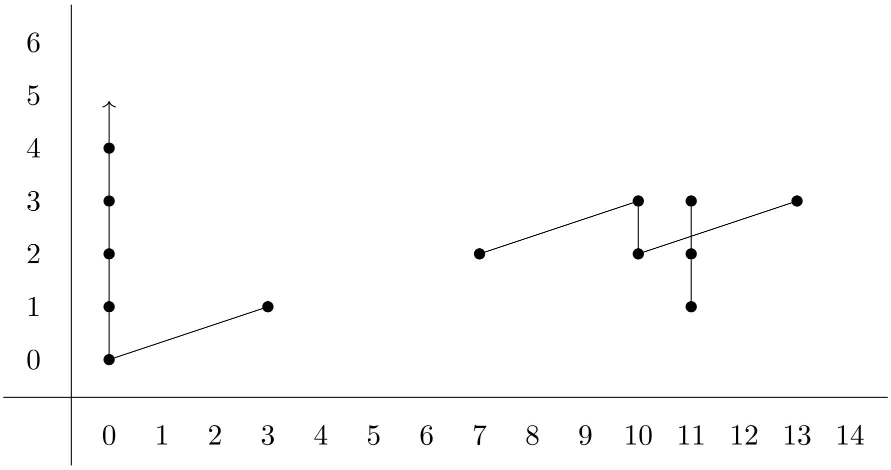


Figure 2.3: $\mathbb{F}_3$-based Adams spectral sequence for $\mathbb{S}_3$

We shall conclude this section with a discussion of the Adams filtration, and how it can help us to understand the charts.

Definition 2.19 ([Nig25] 2.96.)**.** For X a connective spectrum and $E \in \{\mathbb{F}_p, BP\}$ the E-**Adams filtration** on π_*X is given by:

$$F^s\pi_*X := \{\alpha \in \pi_*X \mid \alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_s \text{ where each } \alpha_i \text{ induces 0 on } E\text{-homology}\},$$

and $F^0\pi_*X := \pi_*X$.

By the definition of convergence we obtain that

$$\mathbb{E}_\infty^{s,f} \simeq F^f\pi_sX/F^{f+1}\pi_sX.$$

Further unpacking this we see that the zero-line (that is the line $\mathbb{E}_\infty^{*,0}$) detects classes in $F^0\pi_sX/F^1\pi_sX$, that is precisely classes in the Hurewicz image $\pi_*X \to E_*X$. Similarly the one-line detects classes that can be written as exactly one map that induces 0 on E-homology. Continuing this logic, one observes that the n-line detects classes that can be written as the composition of exactly n maps that induces 0 on E-homology.

2.3 Modified Adams Spectral Sequences

In this section we shall develop the key exploratory technique used in this thesis, both to investigate what is true and to produce proofs.

Since we will work with quotients, we turn to the natural question: *Given a cofibre sequence of spectra:*

$$X \to Y \to Z,$$

Index	0	1	2	3	4	5	6	7	8	9
Group	$\mathbb{Z}_3$	0	0	$\mathbb{F}_3$	0	0	0	$\mathbb{F}_3$	0	0

Table 2.2: $\pi_*\mathbb{S}_3$ in low degrees

when can we compute the $\mathbb{E}_2$-page of the E-Adams spectral sequence of Z from that of X and Y? It turns out this is the case if the fibre sequence yields a short exact sequence

$$0 \to E_*X \to E_*Y \to E_*Z \to 0.$$

This is because we would have a short exact sequence on $\mathbb{E}_1$-pages (as this is essentially the E-homology) and since Ext is a derived functor we get a long exact sequence on $\mathbb{E}_2$-pages. To do better than this we will need to make use of the formalism developed in Section 2.1.

We shall consider our functor

$$Gr : FilSp \to grSp$$

and note that because it is given by a colimit (since colimits in functor categories are computed pointwise) it commutes with colimits. Hence Gr is exact and takes cofibre sequences to cofibre sequences. Now suppose we have a cofibre sequence:

$$X \xrightarrow{f} Y \to Z \tag{2.3}$$

of E-nilpotent complete spectra, then by functoriality we obtain a map

$$\nu(X) \xrightarrow{\nu(f)} \nu(Y)$$

of filtered spectra. It need not hold that $C(\nu(f)) \simeq \nu(Z)$. We shall now show that the spectral sequence given by $C(\nu(f))$ converges to π_*Z regardless and, in the case of quotients by self maps, will have a reasonable $\mathbb{E}_2$-page. There is a functor

$$\text{colim} : FilSp \to Sp$$

which takes a filtered spectrum to its colimit. For E-nilpotent complete spectra we have that $\text{colim} \circ \nu(-) \simeq id$ and hence $\text{colim}(\nu(f)) \simeq f$. Since colimits commute with colimits, we have that applying the colimit functor to the cofibre sequence

$$\nu(X) \xrightarrow{\nu(f)} \nu(Y) \to C(\nu(f))$$

yields the cofibre sequence (2.3) and hence $\text{colim}(C(\nu(f))) \simeq Z$ as desired. We call such a filtered spectrum $C(\nu(f))$ a *modified E-Adams spectral sequence* for Z.

So given a cofibre sequence in Sp, we can produce a cofibre sequence in $FilSp$ by taking the cofibre of the first map. Then one applies Gr, yielding a cofibre sequence in $grSp$ where we know that the cofibre is a modified Adams spectral sequence for the cofibre in spectra. Following Remark 2.37 of [Nig25], since we are indexing our spectral sequences from the second page, the functor

$$\Sigma : FilSp \to FilSp$$

has bidegree (1, -1). We thus obtain a long exact sequence:

$$\cdots \to \mathbb{E}_2^{*,*}(\nu(X)) \to \mathbb{E}_2^{*,*}(\nu(Y)) \to \mathbb{E}_2^{*,*}(C(\nu(f))) \to \mathbb{E}_2^{*-1,*+1}(\nu(X)) \to \cdots$$

Consider the data of X an E-nilpotent complete connective spectrum and $e \in \pi_* X$ such that e lifts to a self map. We produce

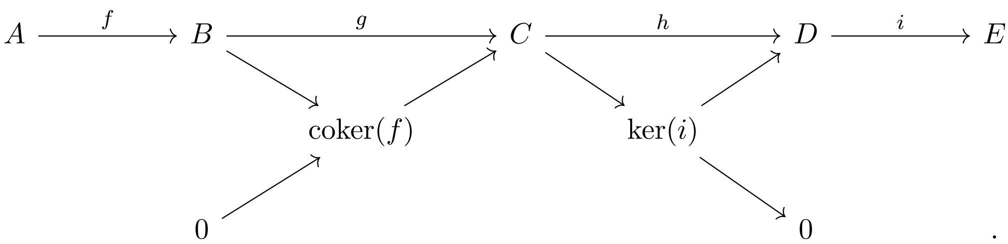
$$\Sigma^{|e|} X \xrightarrow{\times e} X \to X/e.$$

Let $\varepsilon \in \mathbb{E}_2^{*,*}(\nu(X))$ denote the permanent cycle detecting e, it has a stem grading s_ε and a filtration grading f_ε. Then upon passing to Adams spectral sequences we obtain:

$$\cdots \to \mathbb{E}_2^{*-s_\varepsilon,*-f_\varepsilon}(\nu(X)) \xrightarrow{\times\varepsilon} \mathbb{E}_2^{*,*}(\nu(X)) \to \mathbb{E}_2^{*,*}(C(\varepsilon)) \to \mathbb{E}_2^{*-s_\varepsilon-1,*-f_\varepsilon+1}(\nu(X)) \to \cdots \tag{2.4}$$

where $\mathbb{E}_2^{*,*}(C(\varepsilon)) \implies \pi_* X/e$.

From a long exact sequence one may extract short exact sequences by taking appropriate kernels and cokernels:



We can rotate (2.4) so that $\mathbb{E}_2^{*,*}(C(\varepsilon))$ plays the role of C and then produce the short exact sequence

$$0 \to \mathbb{E}_2^{s,f}(\nu(X))/\varepsilon \to \mathbb{E}_2^{s,f}(C(\varepsilon)) \to \mathbb{E}_2^{*-s_\varepsilon-1,*-f_\varepsilon+1}(\nu(X))[\varepsilon] \to 0 \tag{2.5}$$

where $(-)[\varepsilon]$ denotes ε-torsion. Not only does 2.5 allow us to compute $\mathbb{E}_2^{s,f}(C(\varepsilon))$ in degrees where $\mathbb{E}_2^{s,f}(\nu(X))$ is well understood, but by remembering whether a class comes from $\mathbb{E}_2^{s,f}(X)/\varepsilon$ or $\mathbb{E}_2^{*-s_\varepsilon-1,*-f_\varepsilon+1}(\nu(X))[\varepsilon]$ will allow us to rule out the existence of certain multiplicative relations and differentials.

We shall give a worked example of this method and then explain some of its subtleties, computing the $\mathbb{F}_2$-based $\mathbb{E}_2$-page for $\mathbb{S}/2$ (note $\times 2$ is 0 on $\mathbb{F}_2$-homology), so taking Figure 2.2 mod h_0. In Figure 2.4 the classes coming from $\mathbb{E}_2^{s,f}(\nu(\mathbb{S}))/h_0$ will be coloured blue and the classes coming from $\mathbb{E}_2^{*-1,*}(\nu(\mathbb{S}))[h_0]$ will be coloured red. All $\bullet$ are copies of $\mathbb{F}_2$. Just by applying the method described above we compute Figure 2.4, noting we can take differentials and multiplicative relations between two blue classes or two red classes directly from their source.

Remark 2.20. There cannot be a multiplicative relation or a differential from a blue class to a red class.

Let's consider the possibility of an h_0 relation from the blue class ϕ in $(3,1)$ to the red class ψ in $(3,2)$ which a priori could exist. We consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{E}_2^{3,1}(\nu(\mathbb{S}))/h_0 & \xhookrightarrow{f} & \mathbb{E}_2^{3,1}(C(h_0)) & \xrightarrow{g} & \mathbb{E}_2^{2,1}(\nu(\mathbb{S}))[h_0] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times h_0} & & \downarrow{\scriptstyle \times h_0} & & \downarrow{\scriptstyle \times h_0} & & \\
0 & \longrightarrow & \mathbb{E}_2^{3,2}(\nu(\mathbb{S}))/h_0 & \xhookrightarrow{f'} & \mathbb{E}_2^{3,2}(C(h_0)) & \xrightarrow{g'} & \mathbb{E}_2^{2,2}(\nu(\mathbb{S}))[h_0] & \longrightarrow & 0
\end{array}$$

where ψ lifts through g' and ϕ is in the image of f. Since $\mathbb{E}_2^{3,2}(\nu(\mathbb{S}))/h_0 \simeq 0$ (as it is divisible by h_0), the first square commutes and f' is injective we deduce that $h_0\phi = 0$ and since $\psi \neq 0$ we are done. Similar arguments work for other examples as well as differentials instead of multiplicative relations.

However, there can be differentials or multiplicative relations from a red class to a blue class. There is an h_0 relation from the red class ψ in $(2,1)$ to the blue class ϕ in $(2,2)$. To see this we will compute $\pi_2\mathbb{S}/2$ and thereby deduce the relation.

Proposition 2.21. *$\pi_2\mathbb{S}/2 \simeq \mathbb{Z}/4\mathbb{Z}$.*

Proof. We shall present an elementary argument[2], only needing $\mathbb{F}_2$-cohomology and Steenrod squares. We denote the algebra of Steenrod squares $\mathcal{A}_*$ and recall:

- The only nonzero map of spectra $\mathbb{F}_2 \to \mathbb{F}_2[2]$ is Sq^2,
- $\mathcal{A}_1 = \langle Sq^1\rangle, \mathcal{A}_2 = \langle Sq^2\rangle, \mathcal{A}_3 = \langle Sq^3, Sq^2Sq^1\rangle, \mathcal{A}_4 = \langle Sq^4, Sq^3Sq^1\rangle$,
- it follows from the Adem relations that $Sq^2Sq^2 = Sq^3Sq^1$.

From its fibre sequence of definition $\mathbb{S} \xrightarrow{\times 2} \mathbb{S} \to \mathbb{S}/2$ we obtain a short exact sequence

$$0 \to \mathbb{F}_2 \to \pi_2\mathbb{S}/2 \to \mathbb{F}_2 \to 0$$

and our goal is to show this sequence does not split.
By analysing the long exact sequence on $H^* := H^*(-;\mathbb{F}_2)$ we obtain:

- $H^0(\mathbb{S}/2) \simeq \mathbb{F}_2$,
- $H^1(\mathbb{S}/2) \simeq \mathbb{F}_2$,
- $H^n(\mathbb{S}/2) \simeq 0$ for $n \geq 2$.

We shall emulate Serre's approach to unstable homotopy groups and consider the fibre sequence $\tau_{\geq 2}\mathbb{S}/2 \to \mathbb{S}/2 \to \tau_{\leq 1}\mathbb{S}/2$. By analysing the long exact sequence on H^* we obtain:

- $H^*(\tau_{\leq 1}\mathbb{S}/2) \simeq H^*(\mathbb{S}/2)$ for $* \leq 1$,
- $H^{*+1}(\tau_{\leq 1}\mathbb{S}/2) \simeq H^*(\tau_{\geq 2}\mathbb{S}/2)$ for $* \geq 2$.

In particular $H^3(\tau_{\leq 1}\mathbb{S}/2) \simeq H^2(\tau_{\geq 2}\mathbb{S}/2)$.

The key observation that underpins our argument is that $H^2(\tau_{\geq 2}\mathbb{S}/2) \simeq \mathrm{Hom}(\pi_2\mathbb{S}/2, \mathbb{F}_2)$ which follows from the Universal Coefficient Theorem for cohomology since

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_1(\tau_{\geq 2}\mathbb{S}/2), \mathbb{F}_2) \simeq 0,$$

which holds because the Hurewicz Theorem implies $H_1(\tau_{\geq 2}\mathbb{S}/2) \simeq 0$. Hence it suffices to show $H^3(\tau_{\leq 1}\mathbb{S}/2)$ is rank 1 as there is 1 nonzero map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2$ but 3 nonzero maps $\mathbb{F}_2 \oplus \mathbb{F}_2 \to \mathbb{F}_2$.

We shall use the fibre sequence $\tau_{\leq 1}\mathbb{S}/2 \to \mathbb{F}_2 \xrightarrow{j} \mathbb{F}_2[2]$ (where it is clear the cofibre is $\mathbb{F}_2[2]$ by checking the long exact sequence on π_*). We must show j is nonzero, hence implying it is Sq^2: if $j = 0$ then $\tau_{\leq 1}\mathbb{S}/2 \simeq \mathbb{F}_2 \oplus \mathbb{F}_2[1]$ but

$$H^1(\tau_{\leq 1}\mathbb{S}/2) \simeq \mathbb{F}_2 \not\simeq \mathbb{F}_2 \oplus \mathbb{F}_2 \simeq H^1(\mathbb{F}_2 \oplus \mathbb{F}_2[1]).$$

[2]Thank you to Oleksandr Kharchenko for sharing this argument with me.

Now it only remains to compute $H^3(\tau_{\leq 1}\mathbb{S}/2)$, we consider the long exact sequence on H^*:

$$\begin{aligned} &\cdots \to H^1(\mathbb{F}_2) \xrightarrow{Sq^2} H^3(\mathbb{F}_2) \to H^3(\tau_{\leq 1}\mathbb{S}/2) \to H^2(\mathbb{F}_2) \xrightarrow{Sq^2} H^4(\mathbb{F}_2) \to \cdots \\ \simeq &\cdots \to \langle Sq^1 \rangle \xrightarrow{Sq^2} \langle Sq^3, Sq^2Sq^1 \rangle \xrightarrow{f} H^3(\tau_{\leq 1}\mathbb{S}/2) \xrightarrow{g} \langle Sq^2 \rangle \xrightarrow{Sq^2} \langle Sq^4, Sq^3Sq^1 \rangle \to \cdots \end{aligned}$$

We observe the image of f is rank 1 and $g = 0$ yielding that $H^3(\tau_{\leq 1}\mathbb{S}/2) \simeq \mathbb{F}_2$ as desired. □

To examine why this can happen we consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}_2^{2,1}(\nu(\mathbb{S}))/h_0 & \xhookrightarrow{f} & \mathbb{E}_2^{2,1}(C(h_0)) & \xrightarrow{g} & \mathbb{E}_2^{1,1}(\nu(\mathbb{S}))[h_0] & \longrightarrow & 0 \\ & & \downarrow{\times h_0} & & \downarrow{\times h_0} & & \downarrow{\times h_0} & & \\ 0 & \longrightarrow & \mathbb{E}_2^{2,2}(\nu(\mathbb{S}))/h_0 & \xhookrightarrow{f'} & \mathbb{E}_2^{2,2}(C(h_0)) & \xrightarrow{g'} & \mathbb{E}_2^{1,2}(\nu(\mathbb{S}))[h_0] & \longrightarrow & 0 \end{array}$$

where ϕ is in the image of f' and ψ lifts through g. The second commutative square simply does not offer sufficient information for us to rule out $h_0\phi = \psi$, which is in fact the case.

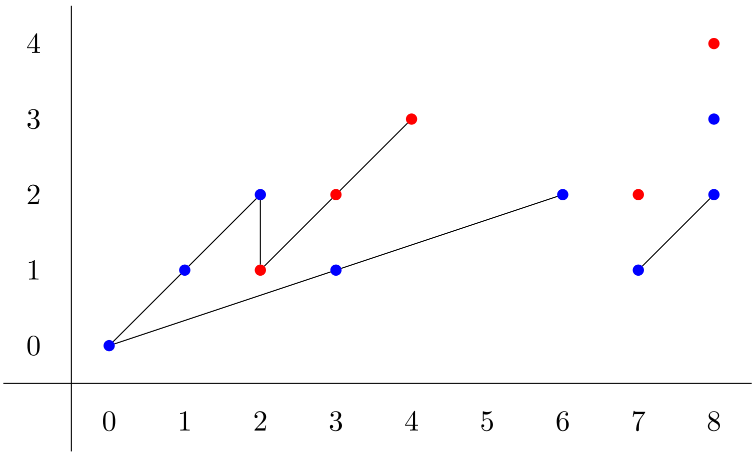


Figure 2.4: $\mathbb{F}_2$-based Adams spectral sequence for $\mathbb{S}/2$

Chapter 3

Detecting v_1-self Maps

In this Chapter we will work at the prime $p = 2$ and everything is implicitly 2-localised. We will show the existence of v_1-self maps for quotients of $\mathbb{S}$, ku and ko.

3.1 Basic properties of v_n-self maps

Showing a spectrum X admits a v_n-self map is carried out in two steps:

1. Show X has a v_n-element, that is an element $\alpha \in \pi_* X$ such that $\alpha \mapsto v_n^j$ along a map $X \to k(1)$ that one needs to construct,
2. show that α lifts to a self map of X.

In the case where X is a ring spectrum the second step is automatic. Next we observe that since $K(n)$ admits a homotopy associative ring structure, the functor

$$K(n)_* : Sp \to grAb$$

naturally refines to a functor

$$K(n)_* : Sp \to gr\mathrm{Mod}_{\mathbb{F}_p[v_n^{\pm}]}$$

and hence we obtain linearity over $\mathbb{F}_p[v_n^{\pm}]$. Hence:

Proposition 3.1. *Suppose $X \xrightarrow{f} Y$ is a map of spectra and X admits a v_n-element α, then Y admits a v_n-element $\pi_* f(\alpha)$ of the same exponent.*

Proof. This follows immediately from linearity over $\mathbb{F}_p[v_n^{\pm}]$. □

We note due to Proposition 1.30 that being of type n is also necessary to admit a v_n-self map.

In order to complete the first step we shall often make use of the BP-based Adams spectral sequence. It follows from its construction that the nth connective Morava K-theory admits a map $BP \xrightarrow{p} k(n)$ such that $\pi_*(p)(v_n) = v_n$. Combining this with what we know about the Adams filtration tells us that the zero-line of the $\mathbb{E}_2$-page of the BP-based Adams spectral sequence will detect v_n-elements in $\pi_* X$.

We shall start with $\mathbb{S}_{(2)}$ and then consider some quotients and closely related spectra. We fix $p = 2$. We have drawn the BP-based Adams spectral sequence for $\mathbb{S}_{(2)}$ in Figure 3.1, where all ∘'s are a copy of $\mathbb{F}_2$, the square in bidegree $(3, 1)$ is a $\mathbb{Z}/4\mathbb{Z}$ and the square in bidgree $(7, 1)$ is a $\mathbb{Z}/16\mathbb{Z}$. The chart is from [IWX19].

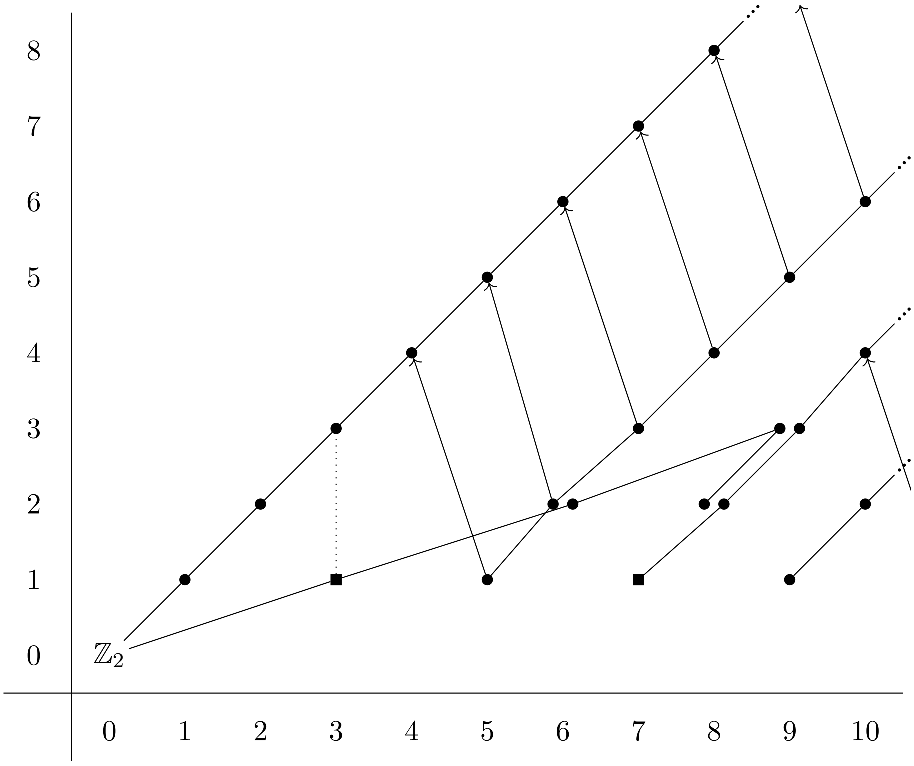


Figure 3.1: BP-based Adams spectral sequence for $\mathbb{S}_{(2)}$ at 2

3.2 The case of $\mathbb{S}_{(2)}$ and $\mathbb{S}/2$

Since $\mathbb{S}_{(2)} \otimes \mathbb{Q} \simeq \mathbb{Q} \not\simeq 0$ we have that $\mathbb{S}_{(2)}$ is of type 0 and hence admits a v_0-self map, which is $\mathbb{S}_{(2)} \xrightarrow{2} \mathbb{S}_{(2)}$.

We consider $\mathbb{S}/2$ which is of type 1 and we shall find its v_1-self map of minimal exponent.

We first tackle step 1: We note that applying BP_* to

$$\mathbb{S} \xrightarrow{\times 2} \mathbb{S} \to \mathbb{S}/2$$

yields a short exact sequence, since the boundary map is zero as $BP_*/2$ is in characteristic 2 and BP_* is in characteristic 0. Hence, in Figure 3.2 we have computed the BP-based Adams spectral sequence for $\mathbb{S}/2$ at 2, using the technique from Section 2.3. We deduce the differential from the red class in bidegree $(4, 0)$ to the blue class in bidegree $(3, 3)$ as $\pi_3\mathbb{S}/2 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and it is the only possible nontrivial differential in this range. As the chart suggests, we have:

Theorem 3.2 ([Rav86] 4.3.2)**.** *The zero-line of Figure 3.2 is* $\mathbb{F}_2[v_1]$.

Thus we have v_1-elements $\tilde{v_1}$ in bidegree $(2, 0)$ and $\tilde{v_1}^4$ in bidegree $(8, 0)$. Clearly $\tilde{v_1}$ is a permanent cycle as it cannot ever support a differential, $\tilde{v_1}^4$ is also a permanent cycle as all possible targets for differentials it could support are killed by or support a d_3. Now we check if $\tilde{v_1}$ lifts to a self map. We note it is the generator of $\pi_2\mathbb{S}/2 \simeq \mathbb{Z}/4\mathbb{Z}$ and we deduce $2\tilde{v_1} = \eta^2$ from Figure 2.4. We apply $[-, \mathbb{S}/2]$ to the cofibre sequence defining $\mathbb{S}/2$ and obtain:

$$\cdots \to [\Sigma^2\mathbb{S}/2, \mathbb{S}/2] \to \pi_2\mathbb{S}/2 \xrightarrow{\times 2} \pi_2\mathbb{S}/2 \to \dots$$
$$\tilde{v_1} \mapsto \eta^2.$$

Thus $\tilde{v_1}$ does not lift to a v_1^1-self map. Next we turn to the class $\tilde{v_1}^4$, and inspect the $8th$ column in Figure 2.4. Since the blue classes did not have an h_0 relation in Figure 2.2 and and by Remark 2.20 there cannot be a relation from a blue class to a red class, we conclude that

$$\pi_8\mathbb{S}/2 \simeq \mathbb{F}_2^{\oplus 3}.$$

We again apply $[-,\mathbb{S}/2]$ to the cofibre sequence defining $\mathbb{S}/2$ and obtain:

$$\begin{aligned} \cdots \to [\Sigma^8\mathbb{S}/2, \mathbb{S}/2] \to \pi_8\mathbb{S}/2 &\xrightarrow{\times 2} \pi_8\mathbb{S}/2 \to \dots \\ \tilde{v_1}^4 &\mapsto 0 \end{aligned}$$

hence the v_1^4-element $\tilde{v_1}^4$ lifts to a self map and thus $\mathbb{S}/2$ admits a v_1^4-self map.

We now remark, the obstruction to admitting a v_1^1-self map is η^2, furthermore, by inspecting Figure 3.2 it is clear that to do better than a v_1^4-self map we would need to kill h_1^3 so that $\tilde{v_1}^2$ in bidegree $(4,0)$ no longer supports a d_3.

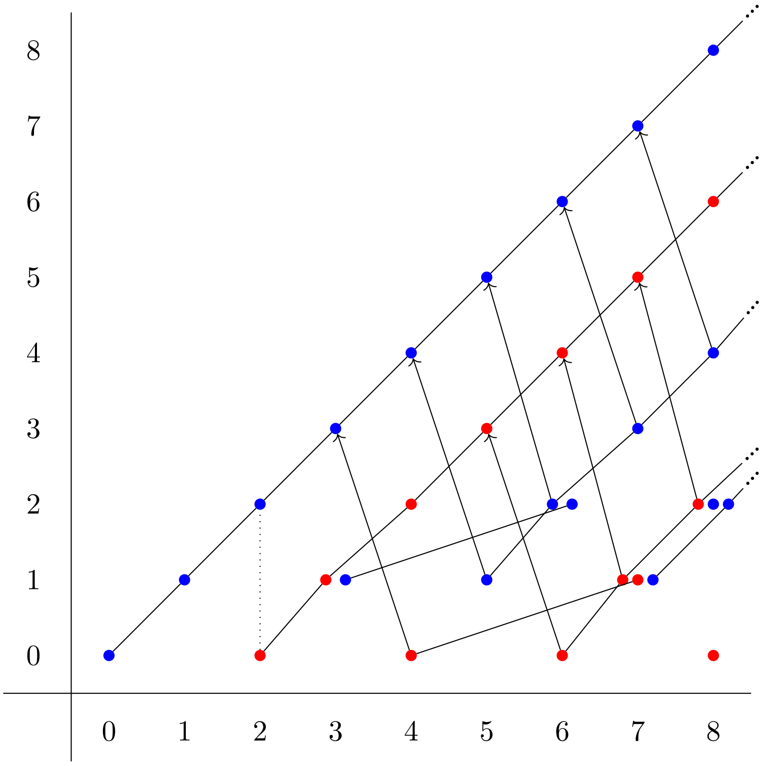


Figure 3.2: BP-based Adams spectral sequence for $\mathbb{S}/2$ at 2

3.3 The case of ku and ko

Before analysing $\mathbb{S}/(2,\eta)$ we shall digress towards ko and $ku := \tau_{\geq 0}KU$. It follows from the Hurewicz theorem that $\pi_0(ku\otimes\mathbb{Q}) \simeq \mathbb{Q}$ and hence ku is of type 0 and has 2 as a v_0^1-self map. Since we are working at $p=2$, we have that $ku/2 \simeq k(1)$ as one can write:

$$ku \simeq \left(\bigotimes_{k\neq 0,1} C(k)\right),$$

using the notation from Proposition 1.15. Hence $ku/2$ admits a v_1^1-element given by v_1 which is also a self map as $k(1)$ admits a multiplicative structure.

We recall that KU admits an action by $\mathbb{Z}/2\mathbb{Z}$ arising from complex conjugation and

$$KO := KU^{h\mathbb{Z}/2\mathbb{Z}} \xrightarrow{c} KO.$$

Now we turn to $ko := \tau_{\geq 0} KO$. We display its BP-based Adams spectral sequence at 2 from [CD25], it is Figure 3.3 where each $\bullet$ is a copy of $\mathbb{F}_2$ and each $\blacksquare$ is a copy of $\mathbb{Z}_{(2)}$.

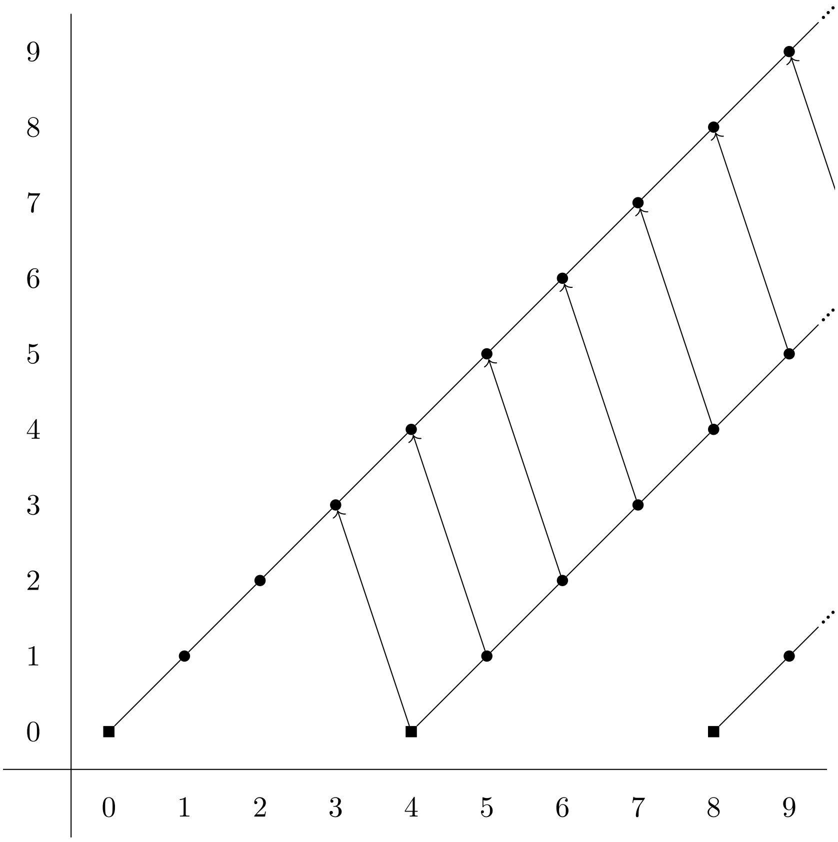


Figure 3.3: BP-based Adams spectral sequence for ko at 2

We can read of Figure 3.3:

$$\pi_* ko \simeq \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \alpha^2 - 4\beta),\ |\eta| = 1, |\alpha| = 4, |\beta| = 8.$$

The class in bidegree $(1,1)$ is the image of h_1 and detects $\eta \in \pi_* ko$. We shall show:

Theorem 3.3. *(Wood) $\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku$ is a cofibre sequence in spectra.*

We will make use of a lemma, originally due to Atiyah, to control what c does and compute Figure 3.3 mod h_1 to ensure the homotopy groups of the target of c are that of ku.

Lemma 3.4 ([CD25] Lemma 4.10.)**.** *The map $c : ko \to ku$ is given on homotopy groups by:*

$$c(\eta) = 0 \qquad c(\alpha) = 2u^2 \qquad c(\beta) = u^4$$

where $u \in \pi_2 ku$ is the generator.

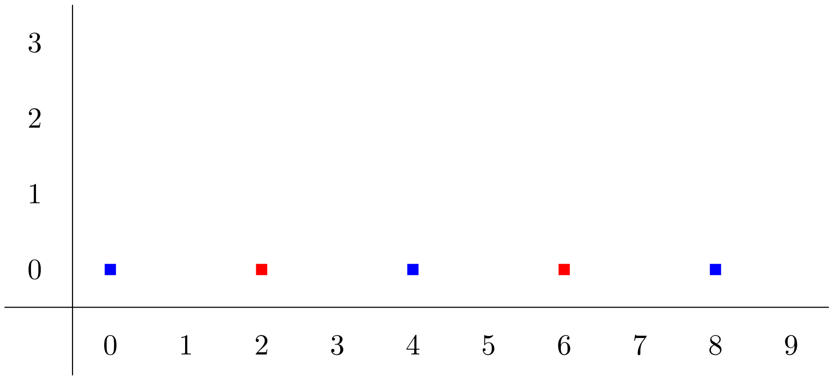


Figure 3.4: Modified BP-based Adams spectral sequence for ko/η at 2

In Figure 3.4 we plot a modified BP-based Adams spectral sequence for ko/η at 2 and hence deduce that $ko/\eta \simeq ku$.

It again follows by the Hurewicz theorem that $\pi_0(ko \otimes \mathbb{Q}) \simeq \mathbb{Q}$ and hence ko is of type 0 with v_0^1-self map given by 2. Now we consider $ko/2$, we shall analyse its homotopy groups, all of which are computed easily via its fibre sequence of definition except $\pi_2 ko/2$. The results of our computations are displayed in Table 3.1.

Proposition 3.5. *$\pi_2 ko/2 \simeq \mathbb{Z}/4\mathbb{Z}$.*

Proof. We consider the following commutative diagram of spectra where the rows are cofibre sequences and $\mathbb{S} \to ko$ is given by the unit map:

$$\begin{array}{ccccc} \mathbb{S} & \xrightarrow{\times 2} & \mathbb{S} & \longrightarrow & \mathbb{S}/2 \\ \downarrow & & \downarrow & & \downarrow \\ ko & \xrightarrow{\times 2} & ko & \longrightarrow & ko/2 \end{array} .$$

We apply π_* and obtain the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \pi_2\mathbb{S} & \xrightarrow{\times 2} & \pi_2\mathbb{S} & \longrightarrow & \pi_2\mathbb{S}/2 & \longrightarrow & \pi_1\mathbb{S} & \xrightarrow{\times 2} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \pi_2 ko & \xrightarrow{\times 2} & \pi_2 ko & \longrightarrow & \pi_2 ko/2 & \longrightarrow & \pi_1 ko & \xrightarrow{\times 2} & \dots \end{array}$$

which is the same as:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\times 2} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \pi_2 ko/2 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\times 2} & \dots \end{array} .$$

Hence we obtain a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \simeq\downarrow & & j\downarrow & & \simeq\downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \pi_2 ko/2 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where the asserted vertical isomorphisms follow from the fact that the unit map $\mathbb{S} \to ko$ sends $\eta \mapsto \eta$ on π_*. Now it follows by the five-lemma that j is an isomorphism.

$\square$

Index mod 8	0	1	2	3	4	5	6	7
Group	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	0	0

Table 3.1: $\pi_* ko/2$

By taking $c : ko \to ku$ mod 2 we obtain a map $\overline{c} : ko/2 \to k(1)$. Inspecting the zero-line of Figure 3.3 and considering what happens when we kill 2, we see there is a candidate v_1^2 element in bidegree (4,0). This class is given in homotopy by α mod 2. However by Lemma 3.4 it follows that $\overline{c}(\alpha) = 0$. Hence we conclude that ko does not have a v_1^2-element since the map $ku_* \to k(1)_*$ is surjective and c comes with the universal property of a limit. Thus ko does not admit a v_1^2-self map which also tells us it does not admit a v_1^1-self map. Using Lemma 3.4 once more we observe that $\overline{c}_*(\beta) = v_1^4$ on π_*. Hence $\beta \in \pi_8 ko/2$ is a v_1^4-element and since it is 2-torsion it is also a self map.

3.4 The case of $\mathbb{S}/(2, \eta)$

Now we shall work out what happens for $\mathbb{S}/(2, \eta)$. We recall the obstruction to $\mathbb{S}/2$ admitting a v_1^1 self map was η^2 as

$$2\tilde{v_1} = \eta^2 \in \pi_2 \mathbb{S}/2.$$

Hence one would expect $\mathbb{S}/(2, \eta)$ to admit a v_1^1-self map. In Figure 3.5 we compute a modified BP-based Adams spectral sequence for $\mathbb{S}/(2, \eta)$ at 2. We see the zero-line is again given by $\mathbb{F}_2[v_1]$ and so we consider the generator $\tilde{v_1} \in \pi_2 \mathbb{S}/(2, \eta)$. Clearly $\tilde{v_1}$ is 2-torsion and hence extends along the quotient map $\Sigma^2 \mathbb{S} \to \Sigma^2 \mathbb{S}/2$ to a map $\Sigma^2 \mathbb{S}/2 \xrightarrow{\overline{v_1}} \mathbb{S}/(2, \eta^2)$, now we need to see that it extends along the cone of η.

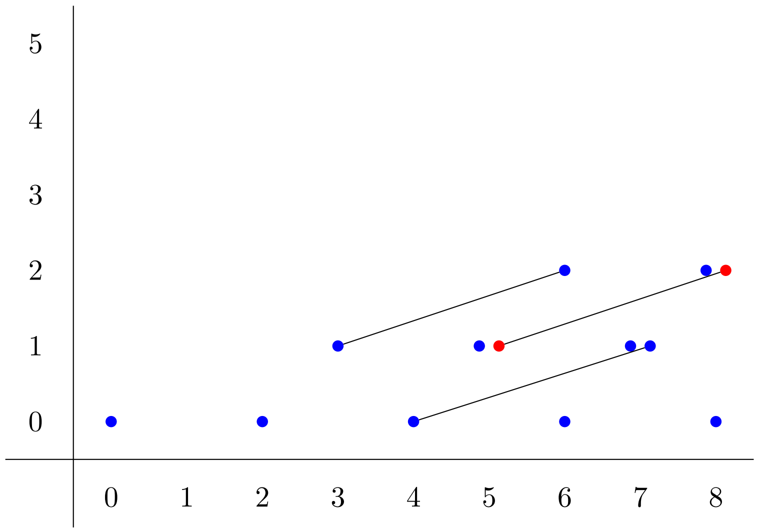


Figure 3.5: Modified BP-based Adams spectral sequence for $\mathbb{S}/(2, \eta)$ at 2

We have:

$$\begin{array}{ccccc}
\Sigma^2\mathbb{S} & \xrightarrow{\times 2} & \Sigma^2\mathbb{S} & \xrightarrow{\tilde{v_1}} & \mathbb{S}/(2,\eta) \\
 & & \downarrow{\scriptstyle p_2} & \nearrow{\scriptstyle \overline{v_1}} & \\
\Sigma^3\mathbb{S}/2 & \xrightarrow{\times \eta} & \Sigma^2\mathbb{S}/2 & & \\
 & & \downarrow{\scriptstyle p_\eta} & & \\
 & & \Sigma^2\mathbb{S}/(2,\eta) & &
\end{array}$$

where p_2 is the cofibre of $\times 2$ and p_η is the cofibre of $\times\eta$, we also use ∂_2 to denote the boundary map for killing 2. We apply $[-, \mathbb{S}/(2,\eta)]$ to produce:

$$\begin{array}{ccccc}
 & & \pi_4\mathbb{S}/(2,\eta) & \xrightarrow{\times \nu} & \pi_7\mathbb{S}/(2,\eta) \\
 & & \downarrow{\scriptstyle \partial_2^*} & & \downarrow{\scriptstyle \partial_2^*} \\
[\Sigma^2\mathbb{S}/2, \mathbb{S}/(2,\eta)] & \xrightarrow{\times \eta} & [\Sigma^3\mathbb{S}/2, \mathbb{S}/(2,\eta)] & \xrightarrow{\times \nu} & [\Sigma^6\mathbb{S}/2, \mathbb{S}/(2,\eta)] \\
\downarrow{\scriptstyle p_2^*} & & \downarrow{\scriptstyle p_2^*} & & \\
\pi_2\mathbb{S}/(2,\eta) & \xrightarrow{\times \eta} & \pi_3\mathbb{S}/(2,\eta) & & .
\end{array}$$

Starting at the top left, must show that $\eta\overline{v_1} = 0$. We shall first show that $p_2^*(\overline{v_1})\eta = 0$ concluding that $\eta\overline{v_1}$ is in the image of ∂_2^*, and then show that it is in fact the image of 0. We will need a relation visible from Figure 2.2, that $\eta\nu = 0$.

We have that $p_2^*(\overline{v_1}) = \tilde{v_1} \in \pi_2\mathbb{S}/(2,\eta)$, and hence we just need to see this class is η-torsion. We consider the blue classes in bidegree $(2,0)$ and $(3,1)$ in Figure 3.5, these come from the red class in bidegree $(2,0)$ and the blue class in bidegree $(3,1)$ in Figure 3.2, respectively. Hence we want to show that the red class in bidegree $(2,0)$ does not divide the blue class in bidegree $(3,1)$ in Figure 3.2. To show this we shall use ν-multiplication. We shall name the generators, a for the homotopy element detected by the red class in $(2,0)$, b for the homotopy element detected by the red class in $(3,1)$ and the blue class in $(3,1)$ detects ν. Now suppose that $\eta a = b + \nu$, then:

$$\begin{aligned}
\nu(b+\nu) &= \nu^2 \\
(\nu\eta)a &= \nu^2 \\
0 &= \nu^2
\end{aligned}$$

which is a contradiction. We conclude $p_2^*(\overline{v_1})\eta = 0$ and by commutativity of the bottom left square that $p_2^*(\eta\overline{v_1}) = 0$, yielding that $\eta\overline{v_1}$ is in the image of ∂_2^*. From Figure 3.5 we see that $\pi_4\mathbb{S}/(2,\eta) \simeq \mathbb{F}_2$. Denote its generator ξ and suppose that $\partial_2^*(\xi) = \eta\overline{v_1}$. We read off Figure 3.5 that ξ supports ν multiplication and that $\nu\xi$ is 2-torsion, and hence $\partial_2^*(\nu\xi) \neq 0$. From commutativity of the top right square we derive that

$$\nu(\eta\overline{v_1}) = \nu\partial_2^*(\xi) \neq 0$$

which contradicts $\nu\eta = 0$. This yields that $\eta\overline{v_1} = 0$, so we conclude that $\overline{v_1}$ does extend over the cone of η. Thus $\mathbb{S}/(2,\eta)$ admits a v_1^1-self map.

3.5 The case of $\mathbb{S}/4$

In this final section, we ponder what happens when one considers $\mathbb{S}/4$. In Figure 3.6 we plot the BP-based Adams spectral sequence for $\mathbb{S}/4$ at 2. The $\bullet$'s are copies of $\mathbb{F}_2$, and the $\blacksquare$'s are copies of $\mathbb{Z}/4\mathbb{Z}$.

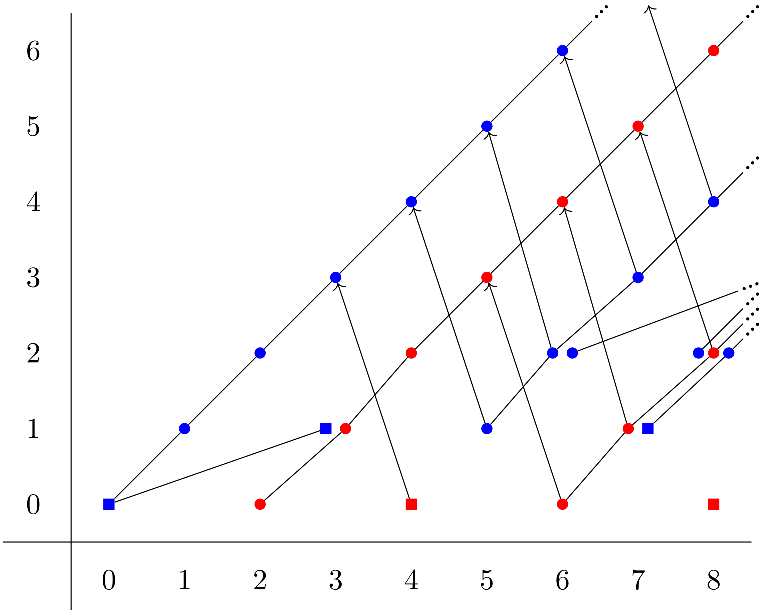


Figure 3.6: BP-based Adams spectral sequence for $\mathbb{S}/4$ at 2

In order to make sense of our spectral sequence data we shall compute some homotopy groups of $\mathbb{S}/4$ using its cofibre sequence of definition as well as the data we already have from the spectral sequence. We display our results in Table 3.2, where we inspected the $\mathbb{F}_2$-based Adams spectral sequence for $\mathbb{S}/4$ at 2 to conclude that the possible extension in π_2 splits.

Index	0	1	2	3	4	5	6	7	8
Group	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{Z}/4\mathbb{Z}$	$\mathbb{F}_2^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$

Table 3.2: $\pi_*\mathbb{S}/4$ in low degrees

We thus deduce the existence of a d_3 from the red square in bidegree $(4, 0)$ to the blue class in bidegree $(3, 3)$, killing it, as well as a hidden extension in stem 4. We shall now encounter new complexities due to working mod 4 instead of mod 2. Looking at Figure 3.6 we see many nonzero classes in the zero-line, however we now note this zero line is detecting the Hurewicz image

$$\mathbb{S}/4 \xrightarrow{h} BP/4$$

where $\pi_*BP/4 \simeq \mathbb{Z}/4[v_1, v_2, \dots]$. Let's consider the homotopy element detected by the red class in bidegree $(2, 0)$, denote it $\tilde{v_1}$. We shall observe that $h(\tilde{v_1}) = 2v_1$ and hence dies along the quotient map $BP/4 \to k(1)$, so even though it is in the correct degree, it is not a v_1-element. We have:

$$\begin{array}{ccc}
\mathbb{S} & \xrightarrow{\times 2} & \mathbb{S} \\
{\scriptstyle \times 4}\downarrow & & \downarrow{\scriptstyle \times 2} \\
\mathbb{S} & \xrightarrow{id} & \mathbb{S} \\
{\scriptstyle p_4}\downarrow & & \downarrow{\scriptstyle p_2} \\
\mathbb{S}/4 & \xrightarrow{j} & \mathbb{S}/2 \\
{\scriptstyle \partial_4}\downarrow & & \downarrow{\scriptstyle \partial_2} \\
\Sigma\mathbb{S} & \xrightarrow{\times 2} & \Sigma\mathbb{S} \\
{\scriptstyle \times 4}\downarrow & & \downarrow{\scriptstyle \times 2} \\
\Sigma\mathbb{S} & \xrightarrow[id]{} & \Sigma\mathbb{S}
\end{array}$$

where the columns are cofibre sequences and the squares commute. Applying π_* we obtain:

$$\begin{array}{cccc}
\tilde{v_1} \in & \pi_2\mathbb{S}/4 & \xrightarrow{j} & \pi_2\mathbb{S}/2 \\
\updownarrow & {\scriptstyle \partial_4}\downarrow & & \downarrow{\scriptstyle \partial_2} \\
\eta \in & \pi_1\mathbb{S} & \xrightarrow{0} & \pi_2\mathbb{S} \\
\downarrow & {\scriptstyle \times 4}\downarrow & & \downarrow{\scriptstyle \times 2} \\
0 & \pi_1\mathbb{S} & \xrightarrow[id]{} & \pi_1\mathbb{S}
\end{array} .$$

From the commutativity of the top square we deduce that $\partial_2(j(\tilde{v_1})) = 0$ where we know that $\mathbb{Z}/4\mathbb{Z} \xrightarrow{\partial_2} \mathbb{F}_2$ is the canonical map killing 2. Hence we deduce $j(\tilde{v_1})$ is either $2v_1$ or 0, since it gets mapped to a nonzero class in BP-homology we conclude $j(\tilde{v_1}) = 2v_1$. We have not produced the v_1-element we are looking for. Next we consider the generator $\tilde{v_1}^2 \in \pi_4\mathbb{S}/4$. By carrying out a similar analysis we see this class dies along the map to $k(1)$. We recall we only have to check powers of 2 and hence consider the homotopy element $\tilde{v_1}^4$ detected by the red square in bidegree $(8,0)$ in Figure 3.6.
Similar to the above we consider:

$$\begin{array}{cccc}
\pi_8\mathbb{S}/4 & \xrightarrow{j} & \pi_8\mathbb{S}/2 & \ni \tilde{v_1}^4 \\
{\scriptstyle \partial_4}\downarrow & & \downarrow{\scriptstyle \partial_2} & \updownarrow \\
\pi_7\mathbb{S} & \xrightarrow{\times 2} & \pi_7\mathbb{S} & \ni 8\sigma \\
{\scriptstyle \times 4}\downarrow & & \downarrow{\scriptstyle \times 2} & \downarrow \\
\pi_7\mathbb{S} & \xrightarrow[id]{} & \pi_7\mathbb{S} & 0
\end{array}$$

and evaluate the groups to get:

$$\begin{array}{cccc}
\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{F}_2^{\oplus 2} & \xrightarrow{j} & \mathbb{F}_2^{\oplus 3} & \ni \tilde{v_1}^4 \\
{\scriptstyle \partial_4}\downarrow & & \downarrow{\scriptstyle \partial_2} & \updownarrow \\
\mathbb{Z}/16\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/16\mathbb{Z} & \ni 8\sigma \\
{\scriptstyle \times 4}\downarrow & & \downarrow{\scriptstyle \times 2} & \downarrow \\
\mathbb{Z}/16\mathbb{Z} & \xrightarrow[id]{} & \mathbb{Z}/16\mathbb{Z} & 0
\end{array} .$$

Since everything in $\pi_8\mathbb{S}/4$ is 4-torsion, ∂_4 maps all of the classes to nonzero classes in $\mathbb{Z}/16\mathbb{Z}$, and that 4σ is in its image. Hence we conclude that 8σ is in the image of $\times 2 \circ \partial_4$.

By commutativity of the top square we conclude that $\tilde{v_1}^4$ is in the image of j, so there is a v_1^4-element in $\pi_8\mathbb{S}/4$. We note that our v_1^4-element need not be unique as lifts need not be unique. Since $\pi_8\mathbb{S}/4$ is all 4-torsion, we know this class lifts to a v_1^4-self map.

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